Financial Frictions and Money-Driven Variability in Velocity

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Abstract. Frictions are introduced in the information structure of a cash-in-advance dynamic stochastic general equilibrium model with the interest of studying the impact of serially correlated monetary shocks on the variability of velocity. Possessing no analytical solution this dynamic environment, a projection method which parameterizes expectations and employs finite elements in the approximation of the system's policy functions is proposed, thoroughly developed and employed on the solution for the equilibrium of the economy. The algorithm is able to efficiently handle the occasionally binding cash-in-advance constraint on transactions, permitting an analysis on the variability of velocity. It is concluded that frictions on the information and financial structure of the economy accentuate a precautionary demand for money balances, increasing the incidence of adjustments on the velocity of transactions as an answer to money growth rate shocks, with the purpose of smoothing consumption.

Key words: finite element method; occasionally binding constraints; cash-in-advance; financial frictions; money velocity

1 Introduction

Money-driven general equilibrium models with great difficulty are capable to explain the variability on velocity observed in the data, due to the insufficient propagation mechanisms associated with monetary shocks. Alas, a proper assessment on the variability of consumption velocity of money is fundamental to understanding the role of money in the business cycle and bestows insight on the microfoundations of monetary policy.

Attempting to answer for a significant allotment of its variability, financial frictions are introduced in the information structure of a cash-in-advance (CIA) dynamic stochastic general equilibrium model with the interest of studying the impact on velocity of serially correlated monetary shocks. On the proposed modeling environment, at each period the agent is required to make its choice on real money holdings prior to the realization of a monetary shock. Uncertainty regarding the current period's realization of the money growth rate incentives the agent to carry additional units of cash than those that would be chosen if instead the portfolio is to be formed ex-post the realization of the shock. Frictions on the information structure accentuate a precautionary demand for money balances, causing variability in velocity. On the model, velocity variability will occur more succinctly at low rates of money growth for consumption smoothing purposes, on these cases the agent chooses to hold money for the next period because of expectations of future low realizations of the growth rate.

Given that this particular dynamic environment posses no analytical solution, a projection method which parameterizes expectations and employs finite elements in its approximations of the policy functions is proposed, thoroughly developed and employed to solve for the equilibrium of the economy. The approximation of the model's functional equations using finite elements is an advantage over other commonly used perturbation or projection methods, such as a parameterization using a Chebyshev polynomial or the linearization of the system around its steady state, because it allows for the fit of numerous low-order polynomials over nonintersecting subdomains of the state space, rather than high-order polynomials over the complete domain. McGrattan (1998), in [10], stresses that the fractioning of the space results in an improvement on the precision of the approximation of the policy functions near regions of the state space that are of higher order or highly nonlinear. Aruoba et. al (2006) in [1] concurs with this result, and finds that finite element approximations proved being the most accurate, stable and of fastest convergence from a considered wide range of projection and perturbation methods.

Following a procedure first introduced by den Haan and Marcet (1990), in [5], the algorithm is able to efficiently handle the CIA constraint on transactions by parameterizing the optimal choice rule on the expectations of future real money holdings. The procedure allows for the inequality constraint on the planner's problem to selectively and occasionally bind, enabling an analysis on the variability of money velocity.

The next section contains a detailed description of the modeling environment and the functional forms of its equilibrium conditions. Section 3 discusses the proposed solution methodology and the implementation of its algorithm. Section 4 shows the business cycle properties of the economy for a benchmark calibration and the implications on velocity of information and financial frictions. Section 5 concludes.

2 Cash-in-Advance Model Economy

The economic environment is modeled after the information structure originally introduced by Svensson (1985), in [12], where a cash-in-advance constrained agent is required to choose money holdings prior to the realization of a stochastic shock to the growth rate of money supply, and further developed by Christiano and Eichenbaum (1992), in [3], where credit goods are incorporated in an ex-ante choice of portfolio holdings. Except for the specification of the information structure, the economic environment is modeled similar to the Cooley and Hansen (1989) CIA model, in [4]. This section presents the planner's problem, its equilibrium conditions, its analytical steady states, and the functional forms for which the solution procedure, discussed in Section 3, involving finite elements and a parameterized expectations algorithm, is implemented.

2.1 Discussion of the economy

In an economy exhibiting information frictions in its future realization of a monetary shock and money is used for transaction purposes, a benevolent social planner maximizes the lifetime utility function of an infinitely lived representative agent by making choices over consumption, labor supply, next period capital, and real money holdings. The consumption good is subject to a CIA constraint, and leisure and capital investment are considered to be credit goods. The timing digression is as follows, the agent begins period t with money M_{t-1} and, at this point, determines its holdings on real money, then receives the lump-sum real transfer T_t, current period's prices are determined and the goods market opens.

Given k_0, the social planner chooses infinite sequences of consumption $\{c_t\}_{t=0}^{\infty}$, labor supply $\{n_t\}_{t=0}^{\infty}$, and real money balances $\{M_{t-1}/p_{t-1}\}_{t=0}^{\infty}$ for an infinitely lived agent, and an infinite sequence of capital stock $\{k_{t+1}\}_{t=0}^{\infty}$ to solve:

$$\max_{\left\{\frac{M_{t-1}}{p_{t-1}}\right\}} E_{t-1}\left\{\max_{\{c_t,n_t,k_{t+1}\}} E_t\left\{\sum_{t=0}^{\infty}\beta^t\left\{\frac{c_t^{1-\tau}-1}{1-\tau}-\gamma_n\frac{n_t^{1+\gamma}}{1+\gamma}\right\}\right\}\right\} \tag{1}$$

subject to the budget constraint:

$$c_t+k_{t+1}+\frac{M_t}{p_{t-1}}=Y_t+(1-\delta)k_t+\frac{M_{t-1}}{p_{t-1}}+T_t \tag{2}$$

and a cash-in-advance constraint:

$$c_t\leq\frac{M_{t-1}}{p_{t-1}}+T_t. \tag{3}$$

Where $Y_t = Ak_t^{\alpha}n_t^{1-\alpha}$, $0<\alpha<1$, A is a production technology parameter, $\beta\in(0,1)$ is the time discount factor, and $\delta\in[0,1]$ is the capital depreciation rate. Investment is defined as the next period's capital stock minus the current period's undepreciated level of capital, i.e. $i_t=k_{t+1}-(1-\delta)k_t$. Changes in the money supply are realized through a lump-sum real transfer T_t to the agent, i.e. $T_t=(M_t-M_{t-1})/p_{t-1}$.

The gross growth rate of money supply θ_t, i.e. $\theta_t=M_t/M_{t-1}$, evolves according to a Markovian process with transitional probabilities Π. Q is the number of possible states of nature of θ, $\sum_{z=1}^{Q}\Pi_{wz}=1$. For each $w=\{1,...,Q\}$ & $r=\{1,...,Q\}$, typical element Π_{wr} is the probability of being on state r on time $t+1$ given the realization of state w in time t:

$$\Pi_{wr}=\Pr\left[\theta_{t+1}=\theta(r)|\theta_t=\theta(w)\right]. \tag{4}$$

The budget constraint in eq. (2) implies that current period's consumption, real money holdings and next period capital stock are financed by actual production, undepreciated capital stock, and the real money balances held from the previous period plus a lump-sum transfer. The CIA constraint requires the agent to hold sufficient real money balances in order to purchase the consumption good.

Defining λ_t as the Lagrangian multiplier associated with the budget constraint and μ_t as the multiplier for the CIA constraint, the planner's problem is identified by the first order conditions in eqs. (5) – (8):

$$\lambda_t = c_t^{-\tau} - \mu_t \tag{5}$$

$$\gamma_n n_t^{\gamma} = \lambda_t (1-\alpha) \frac{Y_t}{n_t} \tag{6}$$

$$\lambda_t = \beta E_t \left\{ \lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{k_{t+1}} + (1-\delta) \right) \right\} \tag{7}$$

$$\frac{E_t \{\lambda_t\}}{p_{t-1}} = \beta E_t \left\{ \frac{c_{t+1}^{-\tau}}{p_t} \right\} \tag{8}$$

the Kuhn-Tucker condition in eq. (9):

$$c_t \leq \left(\frac{M_{t-1}}{p_{t-1}} + T_t \right) \quad \text{and} \quad \mu_t \left[c_t - \left(\frac{M_{t-1}}{p_{t-1}} + T_t \right) \right] = 0 \tag{9}$$

and market clearing conditions for the goods and the money markets, in eqs. (10) – (11), respectively:

$$c_t + k_{t+1} = Y_t + (1-\delta) k_t \tag{10}$$

$$M_t = M_{t-1} + p_{t-1} T_t. \tag{11}$$

Eq. (5) shows the marginal benefit of consumption to be equal to the marginal utility of wealth plus the value of liquidity services needed to finance the transaction. A binding CIA constraint works as a necessary transaction cost which increases the marginal benefit of consumption at the equilibrium allocation. Eq. (6) is the condition for labor market equilibrium, where the marginal cost of labor supplied equals the utility value of its marginal productivity. Eq. (7) is the Euler equation, portraying the relationship between current and expected future consumption decisions; if wealth was to be slightly reduced at the current period and carried over to the next period, the loss of current utility must equal the future discounted value of its real gross return. Equilibrium conditions in eqs. (6) & (7) show how a binding CIA constraint creates a distortion away from consumption (cash goods) towards leisure and capital investment (credit goods). By raising the price of consumption above its production cost, the CIA constraint acts as a tax on consumption, diverting the planner towards acquiring goods which are not subject to this constraint. Eq. (8) indicates that real money holdings are chosen such that ex-ante the realization of the monetary shock, the expected value of holding a unit of money in terms of utility, i.e. $E_t \{\lambda_t\} / p_{t-1}$, equates the next period's expected real marginal benefit of consumption. The Kuhn-Tucker condition in eq. (9) tells that if the constraint

Table 1. Expressions of some endogenous variables

Variable	Expression
Consumption Velocity	$V_t = \frac{C_t}{M_t/p_t}$
Inflation	$\pi_t = \frac{p_t}{p_{t-1}}$
Nominal Interest	$I_t = r_t \cdot E_t \left\{\pi_{t+1}\right\}$
Real Interest	$r_t = E_t \left\{\lambda_{t+1}\left(\alpha \frac{Y_{t+1}}{k_{t+1}} + (1-\delta)\right)\right\}$

is binding, the slackness multiplier is positive, $\mu_t > 0$, and consumption must equal real money holdings, $c_t = M_t/p_{t-1}$; otherwise, if the constraint does not bind, the slackness multiplier is zero, $\mu_t = 0$, and real money holdings may be held for next period, $c_t \leq M_t/p_{t-1}$, in the form of precautionary demand for money. Table 1 indicates the formulae for calculating the consumption velocity of money, inflation, nominal interest, and real interest at time period t. Given that there is no nominal bond in the economy, nominal interest is derived from the Fisher equation.

2.2 Analytical Steady states

Steady states for the CIA model economy are defined by eqs. (12) – (16):

$$\frac{k^{ss}}{y^{ss}} = \frac{\alpha\beta}{1-\beta(1-\delta)} \tag{12}$$

$$\frac{i^{ss}}{y^{ss}} = \delta\frac{k^{ss}}{y^{ss}} \tag{13}$$

$$\frac{c^{ss}}{y^{ss}} = 1 - \frac{i^{ss}}{y^{ss}} \tag{14}$$

$$n^{ss} = \left[\frac{\beta}{\theta^{ss}}\frac{(1-\alpha)}{\gamma_n}\left(\frac{c^{ss}}{k^{ss}}\right)^{-\tau}\left(\frac{y^{ss}}{k^{ss}}\right)^{\frac{\tau-\alpha}{1-\alpha}}\right]^{\frac{1}{\gamma+\tau}} \tag{15}$$

$$\mu^{ss} = \lambda^{ss}\left(\frac{\theta^{ss}}{\beta} - 1\right). \tag{16}$$

The steady state of the CIA constraint is derived from the equilibrium condition of real money balances, eq. (8). Notice the constraint is binding in the steady state for a steady state growth of money supply greater than the time discount factor.

2.3 Equilibrium of the economy

The equilibrium for the CIA economy is denoted by the sequence of real variables $\{\digamma_t\}_{t=0}^{\infty} = \{c_t, n_t, k_{t+1}, M_t/p_{t-1}\}_{t=0}^{\infty}$, given a sequence of monetary growth gross rates $\{\theta_t\}_{t=0}^{\infty}$ evolving according to the transition matrix $\mathit{\Pi}_{wr}$ in

(4), a sequence of money supply and real lump-sum transfers $\{M_t, T_t\}_{t=0}^{\infty}$, and an initial stock of capital k_0, which satisfy the first order conditions (5), (6), (7), & (8), the CIA constraint (9), and the market clearing conditions (10) & (11).

2.4 State space & functional forms of the economy

Θ is the state space of the economy, which can be sub-divided in two subsets; one, Ω, containing the continuous variables of the state space, and a second, Λ, containing the discrete state variables. At time t, the partial state space Ω_t is composed of the possible realizations of the capital stock at time t and the money supply at time $t-2$. Ω has a well defined compact support, i.e. $\Omega = [\underline{\mathrm{k}}, \bar{\mathrm{k}}] \times [\underline{\mathrm{M}}, \bar{\mathrm{M}}]$. Λ_t is composed of the possible realizations of the money growth rate parameters at time t and $t-1$. Λ also has a well defined compact support, i.e. $\Lambda = [\theta(1), ..., \theta(Q)] \times [\theta(1), ..., \theta(Q)]$.

The solution methodology employs the use of time invariant policy functions Υ_{uw} & P_u for all $u = [1, ..., Q]$ and all $w = [1, ..., Q]$, to express the equilibrium conditions of the CIA model economy. The policy function Υ_{uw} is dependent on the past and present realizations of the money growth parameters, while P_u is dependant only on the past realization of the parameter. Conditional on $\theta_{t-1} = \theta(u)$ and $\theta_t = \theta(w)$, $\Upsilon_{uw}(\Omega_t)$ is defined to map the current state of capital and money stock into the control of the conditional expectation function of the Euler equation, i.e. $\Upsilon_{uw}(\Omega_t) \equiv \Upsilon_{\Theta_t} = \Upsilon_w(k_t, M_{t-2}|\theta_{t-1} = \theta(u), \theta_t = \theta(w))$ where $\Upsilon_{\Theta_t} = \beta E_t\{\lambda_{t+1}[\alpha \frac{Y_{t+1}}{k_{uw,t+1}} + 1 - \delta]\}$. Conditional on $\theta_{t-1} = \theta(u)$, $P_u(\Omega_t)$ is defined to map the current state of capital and money stock into the control of the price function, i.e. $P_u(\Omega_t) \equiv p_{\Theta_t} = P_u(k_t, M_{t-2}|\theta_{t-1} = \theta(u))$.

Define $\mathbf{\Upsilon}$ as a $[Q \times Q]$ matrix, where any given row $r \in \{1, ..., Q\}$ contains a vector $\bar{\Upsilon}_r$ of policy functions Υ_{rw}, such that $\mathbf{\Upsilon} = [\bar{\Upsilon}_1, ..., \bar{\Upsilon}_Q]'$. The r^{th} row vector can be represented as $\bar{\Upsilon}_r = [\Upsilon_{r1}, ..., \Upsilon_{rQ}]$. Let $\bar{P}$ be a column vector containing the policy functions P_u for all $u \in \{1, ..., Q\}$, i.e. $\bar{P} = [P_1, ..., P_Q]'$. Using the functional form of the policy functions $\Upsilon_{uw}(k_t, M_{t-2})$ and $P_u(k_t, M_{t-2})$, for all u & w, and the Markovian nature of the exogenous parameters, the residuals of the Euler equation and the equilibrium condition for real money holdings can be written as in eqs. (17) & (18), respectively:

$$R_{uw}^K\left(k, M; \mathbf{\Upsilon}, \bar{P}\right) = \Upsilon_{uw}(k, M) - \beta \sum_{z=1}^{Q} \Pi_{wz}\{\Upsilon_{wz}(\tilde{k}_{uw}, \tilde{M})[\alpha \frac{\tilde{y}_{wz}}{\tilde{k}_{uw}} + 1 - \delta]\} \quad (17)$$

$$R_u^M\left(k, M; \mathbf{\Upsilon}, \bar{P}\right) = \sum_{w=1}^{Q} \Pi_{uw}\left\{\frac{\Upsilon_{uw}(k, M)}{P_u(k, M)} - \sum_{z=1}^{Q} \Pi_{wz}\frac{\tilde{c}_{wz}^{-\tau}}{P_w(\tilde{k}_{uw}, \tilde{M})}\right\}, \quad (18)$$

for all $u = \{1, ..., Q\}$ & $w = \{1, ..., Q\}$. The real variables are defined by

$$y_{uw} = Ak^{\alpha} n_{uw}^{1-\alpha} \quad (19)$$

$$\tilde{k}_{uw} = Ak^{\alpha} n_{uw}^{1-\alpha} + (1-\delta)k - c_{uw} \quad (20)$$

$$n_{uw} = \left[\frac{(1-\alpha)}{\gamma_n} \Upsilon_{uw}(k,M) A k^{\alpha} \right]^{\frac{1-\alpha}{\gamma+1}} \tag{21}$$

$$c_{uw} = \begin{cases} \Upsilon_{uw}(k,M)^{-\tau} & \text{if} \quad \mu_{uw} = 0 \\ M_{+1}/P_u(k,\tilde{M}) & \text{if} \quad \mu_{uw} > 0 \end{cases} . \tag{22}$$

The money supplies ex-ante and ex-post the realization of the growth rate shocks are defined by $\tilde{M}$ and M_{+1}, respectively

$$\tilde{M} = \theta(u) M \tag{23}$$

$$M_{+1} = \theta(w) \tilde{M}. \tag{24}$$

Policy functions in $\mathbf{\Upsilon}$ & $\bar{P}$ are the solutions to $R^K_{uw}(k,M;\mathbf{\Upsilon},\bar{P}) = 0$ for all u & w and $R^M_u(k,M;\mathbf{\Upsilon},\bar{P}) = 0$ for all u, and in combination with y_{uw}, $\tilde{k}_{uw}$, n_{uw}, c_{uw}, $\tilde{M}$ & M_{+1} from eqs. (19) , (20) , (21) , (22), (23) & (24) , and the sequence of money growth rates θ, evolving according the transition matrix Π in (4) , generate the sequences $\{c_t,\, n_t,\, k_{t+1},\, M_t/p_{t-1}\}_{t=0}^{\infty}$, that solve for the equilibrium of this economy along the functional state space Θ.

3 Solution Methodology

The solution procedure involves the usage of finite elements in its approximation of the policy functions and a parameterized expectations algorithm to minimize the weighted absolute value of residual functions $R^K_{uw}(k,M;\mathbf{\Upsilon},\bar{P})$ and $R^M_u(k,M;\mathbf{\Upsilon},\bar{P})$, for all u & w; where the true decision rules $\Upsilon_{uw}(k,M)$ & $P_u(k,M)$ are replaced by the parametric approximations $\upsilon^h_{uw}(k,M)$ & $p^h_u(k,M)$. υ^h_{uw} & p^h_u are approximated using an implementation of the finite element method, that follows that advocated in McGrattan (1996), see [11].

To create the approximate time invariant functions υ^h_{uw} & p^h_u, the space $\Omega = \left[\underline{\mathrm{k}}, \bar{\mathrm{k}}\right] \times \left[\underline{\mathrm{M}}, \bar{\mathrm{M}}\right]$ is divided in n_e nonoverlapping rectangular subdomains called "elements". Parameterization of the policy functions for each element, at each realization of u & w, are constructed using linear combinations of low order polynomials or "basis functions". This procedure creates local approximations for each function. Given the discrete nature of Λ, this state space need not to be divided.

The parameterized functions $\upsilon^h_{uw}(k,M)$ & $p^h_u(k,M)$ are built as follows:

$$\upsilon^h_{uw}(k,M) = \sum_{ij}^{IJ} \upsilon^{uw}_{ij} W_{ij}(k,M) \tag{25}$$

$$p^h_u(k,M) = \sum_{ij}^{IJ} p^u_{ij} W_{ij}(k,M). \tag{26}$$

Where $i = \{1,...,I\}$ indicate capital nodes, $j = \{1,...,J\}$ indicate money supply nodes, $W_{ij}(k,M)$ is a set of linear basis functions dependent on the element

$[k_i, k_{i+1}] \times [M_j, M_{j+1}]$, for all i, j, over which the local approximations are performed. υ_{ij}^{uw} & p_{ij}^{u} are vectors of coefficients to be determined. The parameterized value of the conditional expectation function and the price function over the full state space are obtained by piecing together all the local approximations. The approximate solutions for $\upsilon_{uw}^{h}(k, M)$ & $p_u^h(k, M)$ on Θ are then "piecewise linear functions".

$W_{ij}(k, M)$ are the basis functions that the finite element method employs. These are constructed such that they take a value of zero for most of the space Ω, except for a small interval where they take a simple linear form. The basis functions adopted for these approximations are set such that $W_{ij}(k, M) = \Psi_i(k)\,\Phi_j(M)$, where

$$\Psi_i(k) = \begin{cases} \frac{k-k_{i-1}}{k_i-k_{i-1}} & \text{if } k \in [k_{i-1}, k_i] \\ \frac{k_{i+1}-k}{k_{i+1}-k_i} & \text{if } k \in [k_i, k_{i+1}] \\ 0 & \text{elsewhere} \end{cases} \tag{27}$$

$$\Phi_j(M) = \begin{cases} \frac{M_{-1}-M_{j-1}}{M_j-M_{j-1}} & \text{if } M \in [M_{j-1}, M_j] \\ \frac{M_{j+1}-M}{M_{j+1}-M_j} & \text{if } M \in [M_j, M_{j+1}] \\ 0 & \text{elsewhere} \end{cases}, \tag{28}$$

for all $i,\ j$. $\Psi_i(k)$ & $\Phi_j(M)$ have the shape of a continuous pyramid which peaks at nodal points $k = k_i$ & $M = M_j$, respectively, and are only non-zero on the surrounding elements of these nodes.

The approximations $\upsilon_{uw}^{h}(k, M)$ & $p_u^h(k, M)$ are chosen to simultaneously satisfy the equations:

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} \omega(k, M)\, R_{uw}^{K}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0 \tag{29}$$

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} \omega(k, M)\, R_{u}^{M}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0, \tag{30}$$

for all $u = \{1, ..., Q\}$ & $w = \{1, ..., Q\}$. $\omega(k, M)$ is a weighting function, and $R_{uw}^{K}(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h)$ & $R_{u}^{M}(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h)$ are the residuals from the Euler equation and the equilibrium condition for real money holdings, where the true policy functions $\boldsymbol{\Upsilon}$ & $\bar{P}$ are replaced by the vectors of parametric approximations $\boldsymbol{\upsilon}^h$ & $\bar{p}^h$. A Galerkin scheme is employed to find the vectors of coefficients υ_{ij}^{uw} & p_{ij}^{u}, for all i, j and all u & w, which solves for the weighted residual equations (29) & (30) over the complete space Θ. The Galerkin scheme employs the basis functions $W_{ij}(k, M)$ as weights for $R_{uw}^{K}(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h)$ & $R_{u}^{M}(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h)$:

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} W_{ij}(k, M)\, R_{uw}^{K}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0 \tag{31}$$

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} W_{ij}(k, M)\, R_{u}^{M}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0 \tag{32}$$

for all $i,\ j$ and all states u & w. Since the basis functions are only nonzero surrounding their nodes, eqs. (31) & (32) can be rewritten in terms of the individual elements:

$$\sum_{e=1}^{n_e} \int_{\Omega_e} W_{ij}\left(k, M\right) R^K_{uw}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0 \tag{33}$$

$$\sum_{e=1}^{n_e} \int_{\Omega_e} W_{ij}\left(k, M\right) R^M_{u}\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right) dkdM = 0, \tag{34}$$

for all $i,\ j$ and all states u & w. n_e is the total number of elements and Ω_e is the capital and money stock domain covered by the element e.

A Newton algorithm is used to find the coefficients for $[\boldsymbol{\upsilon}_s, \bar{p}_s]$ which solve for the nonlinear system of equations H:

$$H\left([\boldsymbol{\upsilon}_s, \bar{p}_s]\right) = 0. \tag{35}$$

Where $H\left([\boldsymbol{\upsilon}_s, \bar{p}_s]\right)$ is denoted by eqs. (33) & (34). The first step is to choose initial vectors of coefficients $[\boldsymbol{\upsilon}_{s_0}, \bar{p}_{s_0}]$, and iterate as follows:

$$\left[\boldsymbol{\upsilon}_{s_{l+1}}, \bar{p}_{s_{l+1}}\right] = [\boldsymbol{\upsilon}_{s_l}, \bar{p}_{s_l}] - J\left([\boldsymbol{\upsilon}_{s_l}, \bar{p}_{s_l}]\right)^{-1} H\left([\boldsymbol{\upsilon}_{s_l}, \bar{p}_{u_l}]\right). \tag{36}$$

J is the Jacobian matrix of H, and $[\boldsymbol{\upsilon}_{s_l}, \bar{p}_{s_l}]$ is the l^{th} iteration of $[\boldsymbol{\upsilon}_s, \bar{p}_s]$. The algorithm solves for the parameterized version of the conditional expectation function of the Euler equation $\upsilon^h_{uw}\left(k, M\right)$ and for the price function $p^h_u\left(k, M\right)$, for $\{\theta\left(u\right), \theta\left(w\right)\} \subset \Lambda$ and $\{k, M\} \subset \Omega$. Convergence is assumed to have occurred once $\left\|\left[\boldsymbol{\upsilon}_{s_{l+1}}, \bar{p}_{s_{l+1}}\right] - [\boldsymbol{\upsilon}_{s_l}, \bar{p}_{s_l}]\right\| < 10^{-7}$.

3.1 Algorithm Chronology

The following steps summarize the chronology of the parameterized expectations algorithm, which employs finite elements in the approximation of the policy functions used to solve the stochastic model:

1. Choose the location and quantity I & J of nodes along the capital and money supply domain (i.e. k_i & M_j for $i = \{1, ..., I\}$, $j = \{1, ..., J\}$), which will delimit the n_e nonoverlapping elements in Ω, and use a Gaussian-Legendre quadrature rule to identify abscissas and weights of capital and money stock on each element.
2. Identify the Q states of the Markovian monetary shock and the transition matrix Π_{wr} of the process.
3. For each realization $\theta_{t-1} = \theta\left(u\right)$ approximate P_u with p^h_u using (26), and for each $\theta_t = \theta\left(w\right)$ approximate Υ_{uw} with υ^h_{uw} using (25), for all Gauss-Legendre capital and money stock abscissas on each element along the state space Ω_t.
4. Initiate a recursive solution procedure creating a conjecture that the CIA constraint for this economy, in eq. (9), does not bind, i.e. $\mu_{uw} = 0$. Consumption is automatically solved from eq. (22): $c_{uw} = \left[\upsilon^h_{uw}\left(\Omega_t\right)\right]^{-\tau}$.

Table 2. Calibration parameters of the benchmark economy

A	α	β	δ	γ	τ	θ^{ss}
1.00	0.33	0.99	0.02	0.00	3.00	1.01727

5. Compute the values of employment n_{uw}, actual output y_{uw}, next period capital $\tilde{k}_{uw}$, using eqs. (21), (19) and (20) respectively, and of real money holdings $M_{+1}/p_u^h(\Omega_t)$. Where $M_{+1} = \theta(w) \cdot \tilde{M}$, and $\tilde{M} = \theta(u) \cdot M$.
6. Check whether the initial conjecture was correct. If "yes", go to next step. If not, the constraint binds: $\mu_{uw} > 0$ & $c_{uw} = M_{+1}/p_u^h(\Omega_t)$. The value of the CIA multiplier becomes $\mu_{uw} = [c_{uw}^{-\tau} - \upsilon_{uw}^h(\Omega_t)]$.
7. For each realization $\theta_t = \theta(w)$ approximate P_w with p_w^h, and for each $\theta_{t+1} = \theta(z)$ approximate Υ_{wz} with υ_{wz}^h, for all Gauss-Legendre capital and money stock abscissas of each element along the state space Ω_{t+1}.
8. Create a conjecture that the CIA constraint does not bind at time $t+1$, i.e. $\mu_{wz} = 0$. Consumption becomes $c_{wz} = \left[\upsilon_{wz}^h(\Omega_{t+1})\right]^{-\tau}$.
9. Compute the value of each possible n_{wz}, y_{wz}, and $\tilde{M}_{+1}/p_w^h(\Omega_{t+1})$. Where $\tilde{M}_{+1} = \theta(z) \cdot M_{+1}$.
10. Check whether the conjecture in Step 8 was correct. If "yes", go to next step. If not, the constraint binds: $\mu_{wz} > 0$, & $c_{wz} = \tilde{M}_{+1}/p_w^h(\Omega_{t+1})$. The value of the CIA multiplier becomes $\mu_{wz} = [c_{wz}^{-\tau} - \upsilon_{wz}^h(\Omega_{t+1})]$.
11. Construct the residual functions $R_{uw}^K\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right)$ & $R_u^M\left(k, M; \boldsymbol{\upsilon}^h, \bar{p}^h\right)$ using eqs. (17) & (18) for each state u & w.
12. If the weighted approximations of the residual functions for each element, in eqs. (33) & (34), are sufficiently close to zero then "stop", else update the vectors of coefficients υ_{ij}^{uw} & p_{ij}^u, for all u & w, as in (36) and go to Step 3.

4 Business Cycle Properties

4.1 Calibration

When calibrating the model, the time interval is consistent with that of a quarter of a year. The time discount factor is set to 0.99, depreciation rate at 0.019, the capital elasticity of output to 1/3, and the inverse of labor supply elasticity is set to 0, denoting Hansen's indivisible labor (see [7]), and the steady state labor supply is set to 0.31. Steady state inflation rate is set to the U.S. quarterly data mean value from 1959:III to 1998:III, that is 1.727%. This calibration of parameters is comparable to that of Walsh (1998), in [14]. For the benchmark calibration, following Kocherlakota (1996) in [9], the intertemporal elasticity of substitution in consumption is set to equal 1/3, yielding $\tau = 3.00$. These parameters are summarized in Table 2.

The growth rate of the money supply θ_t is assumed to evolve according to the process in eq. (37) :

$$\theta_{t+1} = (1-\rho)\theta^{ss} + \rho\theta_t + \varepsilon_{t+1} \quad where\ \varepsilon \sim N\left(0, \sigma^2\right). \tag{37}$$

Table 3. Steady states of the benchmark economy

c^{ss}	k^{ss}	n^{ss}	Y^{ss}	μ^{ss}	π^{ss}	r^{ss}	I^{ss}	V^{ss}
0.82	11.94	0.31	1.04	0.05	1.017	1.01	1.03	1.00

A first order autoregressive estimation of M2, using quarterly data from 1959:III to 1998:III, yields $\rho = 0.727$ and $\sigma_\varepsilon = 0.01$. The money growth AR(1) process is approximated by a discrete Markov chain using the methodology advanced by Tauchen and Hussey (1991), see [13]. Also, given the high persistence of the process, an adjustment to the weighting function recommended by Flodén (2008), in [6], is performed. Five states for the money supply growth rate are considered; these return the state vector θ = [0.9846, 1.0017, 1.0173, 1.0328, 1.0500], and the state transition matrix in (38).

$$\Pi = \begin{bmatrix} 0.51136 & 0.45235 & 0.03601 & 0.00028 & 0.00000 \\ 0.07677 & 0.58264 & 0.32302 & 0.01751 & 0.00005 \\ 0.00361 & 0.19100 & 0.61076 & 0.19100 & 0.00361 \\ 0.00005 & 0.01751 & 0.32302 & 0.58264 & 0.07677 \\ 0.00000 & 0.00028 & 0.03601 & 0.45235 & 0.51136 \end{bmatrix}. \tag{38}$$

4.2 Model Economy Steady States

Finding the steady state values of the endogenous variables of the model requires the calculation of $\upsilon^h_{uw}(k, M)$ and $p^h_u(k, M)$ for a determinate environment, and finding k such that $k = \tilde{k}_{uw}$, and the realization of the steady state ratios contained in eqs. (12) − (16). These objectives are achieved by minimizing the geometric distance between the vector $\bar{x}^{ss}$, containing the values of the analytical steady state and the vector $\bar{x}^h_{uw}$, containing the steady state approximations. Money growth is fixed at $\theta(u) = \theta(w) = \theta(z) = \theta^{ss}$. Vectors $\bar{x}^{ss}$ & $\bar{x}^h_{uw}$ are defined in eqs. (39) & (40), respectively, and the geometric distance is defined in eq. (41). Table 3 presents the steady state values for the endogenous variables of the economy at the benchmark calibration; as expected, $I = \pi/\beta$ and the CIA constraint is binding at the steady state.

$$\bar{x}^{ss} = \left[k^{ss}, \frac{k^{ss}}{y^{ss}}, \frac{i^{ss}}{y^{ss}}, \frac{c^{ss}}{y^{ss}}, n^{ss}, \mu^{ss}\right]' \tag{39}$$

$$\bar{x}^h_{uw} = \left[\tilde{k}_{uw}, \frac{k}{y_{uw}}, \frac{\tilde{k}_{uw} - (1-\delta)\,k}{y_{uw}}, \frac{c_{uw}}{y_{uw}}, n_{uw}, c^{-\tau}_{uw} - \Upsilon_{uw}(k, M)\right]' \tag{40}$$

$$\left\|\bar{x}^{ss} - \bar{x}^h_{uw}\right\| = \sqrt{\sum_{i=1}^{6}\left(x^{ss}_i - x^h_{uw,i}\right)^2}. \tag{41}$$

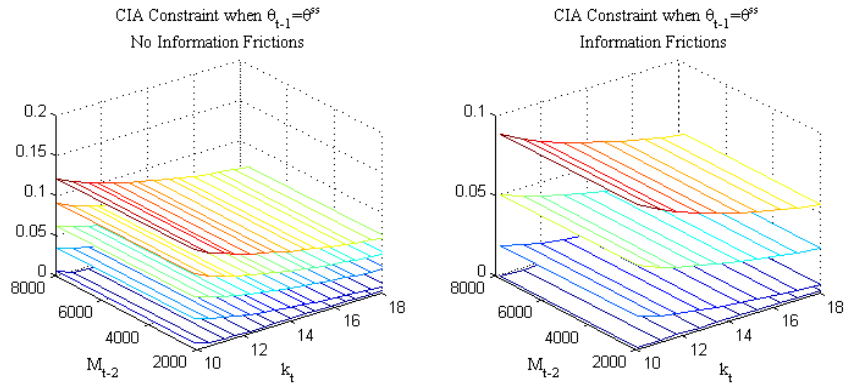

Fig. 1. The left and right panels illustrate the values assumed by μ_t on the benchmark economy when frictions are absent and present on the information structure, respectively. On the environment with frictions, the CIA-constraint binds at the steady state money growth rate but it does not at lesser growth rates.

4.3 Velocity under a serially correlated money growth rate

Velocity differs from one when the agent's real money holdings on a period are greater than the expenditures on consumption, and will stay constant at one when the choice of investing in an interest bearing venture, such as capital, is preferred to holding additional units of cash. The latter occurs when real interest rate is sufficiently high and the variation of expected marginal utility of consumption is relatively small, see Hodrick R. et. al. (1991) in [8].

Fig. 1 contains two comparable cases that illustrate the effects of financial frictions on the variability of velocity. The left panel of Fig. 1 considers the scenario where frictions are not present in the information structure of the benchmark economy and the right panel the scenario where they are. Each panel contains a mesh depicting the level values of μ_t over the state space Ω_t for each possible realization of θ_t, given that $\theta_{t-1} = \theta^{ss}$. μ_t increases with θ_t. Considering first the frictionless environment, μ_t is positive and velocity is equal to one on almost the totality of the state space Θ_t. The CIA constraint binds at the steady state level of capital stock on all but the lowest realization of the money growth parameter. Serially correlated money growth shocks, on a frictionless environment, can only yield variable velocity at sufficiently low money growth rates combined with sufficiently low real interest rates. It is possible to notice that below a critical level of capital stock, the real interest is sufficiently high to dissuade the agent from holding any money for precautionary reasons, and instead acquire interest bearing physical capital in its portfolio. Cao-Alvira (2009), in [2], presents a thorough description of a projection methodology that solves for a similar cash-in-advance model economy with no frictions in its information structure, as well as a robustness analysis and cutoff points for the existence of variability on velocity.

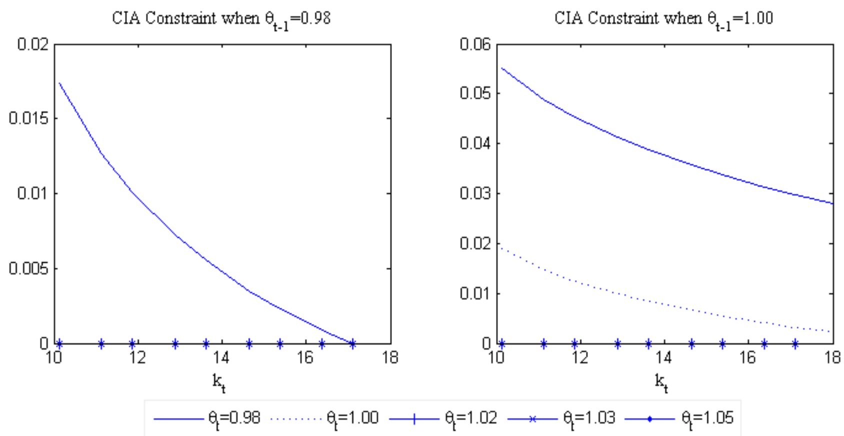

Fig. 2. The two panels depict those cases where the previously realized growth parameter is greater than the steady state value. The lesser the realization of the previous period's money growth parameter, the more possible cases will occur that the agent will decide to hold on cash for next period and the CIA-constraint will not bind.

Observing the case depicted in panel 2 of Fig. 1, the environment where frictions are present, it is possible to notice two key features. First, μ_t is positive, or the CIA constraint binds, at the steady state of the benchmark economy. This result is consistent with those findings reported in Table 3 and, as expected, information or financial frictions do not affect the steady state specification of the model. And second, μ_t equals zero, on all considered values of Ω_t, for those parameters $\theta_t < \theta^{ss}$. Serially correlated shocks on monetary growth, combined with frictions on the information structure, pushes the agent to accumulate excessive real money holdings on states of nature that exhibit a lower than average money growth rate. On these states, cash is being transferred from one period to the next in order to smooth consumption and decrease its expected marginal utility variability.

The results depicted on Fig. 2 and Fig. 3 better assist on illustrating this feature of the environment containing frictions. Each figure contains two panels where the value of μ_t is depicted for all considered values of the capital stock and the money growth parameters, given a fixed value of M_{t-2} and a previous realization of the growth parameter. The panels on Fig. 2 depict those cases where the previously realized growth parameter is lesser than the steady state value, and the panels in Fig. 3 depict the cases where these parameters are greater than the steady state value. As can be observed, the lesser the previous period's monetary growth, the more possible scenarios will occur that the agent decides to hold on cash for the future. Examining both panels on Fig. 2, it is worth noticing that agents will decide to carry money on to the next period even when the current realization of monetary growth is at the steady state level, if the previous value of the parameter is less than the steady state. Only on the

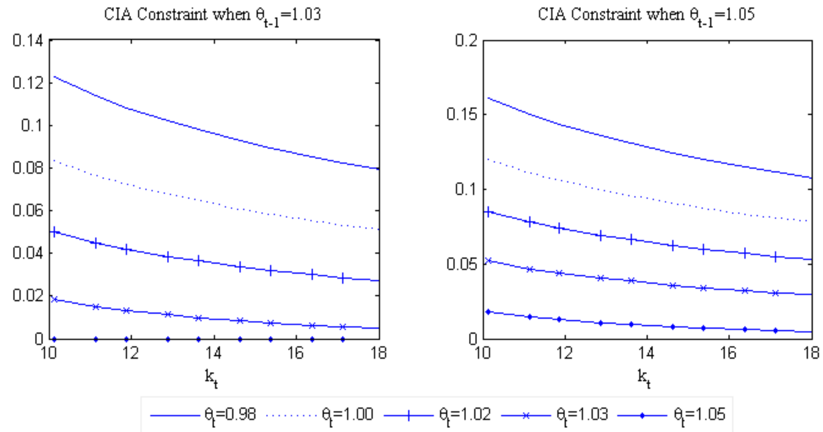

Fig. 3. The two panels depict those cases where the previously realized growth parameter is greater than the steady state value. Only on the scenario where previously the highest possible growth parameter is realized, the right panel, the agent will decide to never carry excessive cash holdings and the CIA constraint will always bind.

scenario where the highest possible growth parameter is realized in the previous period, the right panel of Fig. 3, the agent will decide to never carry excessive cash holdings.

Table 4 documents the simulated moments of the friction and frictionless version of the benchmark economy and the sample moments of the US economy for the time period 1959:III to 1998:III of three economic indicators mostly used in the literature to measure the variability of velocity. These are the coefficient of variation of velocity, measured as $cv(V) = (\sigma_V/\mu_V) \cdot 100$, the correlation between velocity and gross consumption growth, where consumption growth is defined as the ratio between current and past consumption, and the correlation of inflation and real interest rate. While the first index is a straightforward measure of variability, the last two are good indicators on how monetary variables affect the real economy.

Comparing the fit of both simulated modeling environments, that with the richest information structure represents a serious improvement in performance. On a setting which allows for frictions, money shocks are able to explain a fourth of the observed variability in velocity, while a frictionless environment can explain only 1.5% of it. When considering the correlation between velocity and consumption growth, the model containing frictions is able to explain its full comovement, while the frictionless model is able to explain only half of it. The US economy sample value for this estimate has a negative sign, which both simulated models can replicate. The reasoning for this result is that on high realizations of the money growth parameter, due to inflation, the purchasing power of money decreases, causing the need to increase the velocity of balances, and households choose to decrease consumption in order to smooth it with that

Table 4. Simulated moments of three key indexes for the friction and frictionless version of the benchmark economy and those for the US economy for the time period 1959:III to 1998:III. Numbers in parenthesis are standard deviations over 500 simulations. US economy values as reported by Wang W., Shi S. (2006).

Indexes	Benchmark	Information Friction	Data
$cv\,(V)$	0.0306	0.4600	1.9719
	(0.0127)	(0.0480)	
$corr\left(V, \frac{C_t}{C_{t-1}}\right)$	-0.1534	-0.4072	-0.3537
	(0.0663)	(0.073)	
$corr\,(\pi, I)$	0.7300	0.0102	0.5135
	(0.0413)	(0.0268)	

of previous periods. The inability of a standard cash-in-advance environment to explain the amount of correlation between velocity and consumption growth is well documented in the literature and, as in this paper, some researchers on velocity have parted from the original formulation of the model to be able to replicate it; some with better luck than others. A nonexhaustive alphabetically ordered list includes Hodrick R. et. al (1991), in. [8], Svensson L. (1985), in [12], and Wang W., Shi S. (2006), in [15].

As can be observed in the simulated values for the correlation of inflation and real interest rate, the cost of improving the fit on velocity by considering frictions is a degraded specification of inflation on the calibration. The frictions in the information structure over smooths inflation decreasing its variability, compared to the frictionless case.

5 Conclusion

This paper introduces a modeling environment where financial frictions are incorporated on the selection of assets on an agent's portfolio with the interest of studying the impact of serially correlated monetary shocks on the variability of velocity. The solution technique proposed and developed in finding the equilibrium of the system employs the usage of finite elements in the approximation of the policy functions, and a parameterization of the agent's expectations over the choice of future real money balances. The method converges with rapid speed and is efficient approximating the studied highly nonlinear environment. Because the algorithm allows for the cash-in-advance constraint on consumption expenditures to selectively bind, instead of exogenously assuming it always does as other more conventional methods require, it is possible for the equilibrium conditions to achieve variability in the velocity of money holdings. By considering frictions on the financial structure of the economy, a solely money-driven monetary benchmark economy is able to explain over a fourth of the observed variability on the data, representing a considerable improvement over frictionless environments.

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