Note on Velocity: Solution to a cash-in-advance economy using finite elements and a parameterized expectations algorithm

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Abstract

A solution methodology employing finite elements and a parameterized expectations algorithm is proposed for computing the equilibrium of a CIA model economy. The stochastic growth model considered introduces money via a CIA constraint, and takes consumption as the cash good and leisure and capital investment as credit goods.

The ability of the methodology to solve the model while allowing the CIA constraint to occasionally bind permits us to analyze money velocity variability; for which a note is included. Velocity volatility arises due to uncertainty on future realizations of the state of the economy and increases with the agent's risk aversion.

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Keywords: finite element method; occasionally binding constraints; cash-in-advance; money velocity

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1 Introduction

A parameterized expectations algorithm, employing a finite element method, is developed for computing the equilibrium of a cash-in-advance model economy. A cash-in-advance economy is chosen for the implementation of the proposed methodology due to its wide usage by practitioners as a means of introducing money in the optimal growth model. By parameterizing the expectations of the future optimal choice of real money holdings, following the procedure advocated in Christiano and Fisher (2000), the algorithm is able to efficiently handle the cash-in-advance constraint for transactions. Allowing the inequality constraint to occasionally bind, the results of the solution methodology enable an analysis on the variability of velocity of money; a note on the matter is included in the paper.

A formal introduction to the implementation of finite element methods for the computation of equilibrium of optimal growth models (OGM, hereafter) is found in McGrattan (1996). In this implementation the functional equation of the OGM is solved by subdividing the domain of the state space into nonintersecting subdomains called "elements", and fitting low-order polynomials on each subdomain rather than high-order polynomials over the complete domain. McGrattan (1999) stresses the superiority of finite element methods or local approximations over global approximations when parameterizing a policy function near regions of the state space that are of higher order or highly nonlinear. Aruoba et. al (2006) provides a comprehensive survey of various perturbation and projection methods, logarithmic linearizations included in the former and finite elements in the latter, applied as solution procedures to optimal growth models with leisure, and compares their performances. Among its findings it is that finite elements proved being, over a full range of parameterizations, the most accurate, stable and with fastest converging of all methods. The authors also found that accuracy, speed and simplicity are well achieved by perturbation methods as long as the analysis is restricted to a domain near the steady state of the economy. Aruoba et. al (2006) does not consider inequality constraints due to the difficulties experienced by perturbation methods addressing these highly nonlinear environments.

The framework of the dynamic stochastic economy chosen for the application of the proposed solution methodology is similar to that of Cooley and Hansen (1989). In this environment, money is introduced as a means of exchange, via a cash-in advance constraint, and a consumption good is produced by a combination of labor and capital and can only be purchased with real money holdings. Consumption is then considered a cash good, and leisure and capital investment as credit goods. The growth rate of the monetary supply is assumed to follow a Markov process. For a benchmark economy, the CIA constraint is found to not bind on low realizations of the serially correlated parameter denoting money growth, combined with a sufficiently high accumulation of capital. This finding is consistent with the existence of a precautionary demand for real money balances, intending to smooth consumption. An agent will hold excessive real money balances when expecting a low realization of money growth in the future, and a low expected return on the real asset.

The note on velocity variability intends to shed light on the velocity implications of a modeling environment that has become a workhorse for monetary optimal growth literature. On this note, simulation results for various model calibrations are presented and discussed. A finding reported is the tendency of the model to predict a higher variability of velocity for a more risk averse agent, measured by the intertemporal elasticity of substitution of consumption. A finding also reported in Hodrick et al. (1991).

To a specific calibration of the model economy is given particular attention due to its regular usage by practitioners, that is the specification of the agent's utility providing a logarithmic form on consumption and Hansen (1985) indivisibility of labor. On this specification of the utility function the CIA constraint is found to always bind for all the considered states of the economy. This result is significant because it benchmarks a calibration for a model economy where the employment of a solution methodology that imposes a value for velocity always equal to one could be considered appropriate.

Realistic parameterizations of a Cooley and Hansen (1989) CIA setting does not provide acceptable velocity variability. The benchmark economy presented in this paper can account for less than two percent of the variability observed on the data. Departures from the standard formulations of the CIA economies of Cooley and Hansen (1989), Lucas (1982) and Lucas and Stockey (1987) have been considered to reconcile the velocity predictions of the model with those observed in the data. A nonexhaustive list of these departures includes the addition of frictions in the timing structure of information entering the economy, as in Svensson (1985), allowing habit formation on the agent's preferences, as does Hodrick et. al. (1991), the inclusion that appears on Hromcová (2008) of intermediation cost which causes costly credit, and the imposition of costly search in the goods and the labor markets introduced by Shi (2006). The solution methodology proposed in this paper could be modified to include similar modeling environments to the ones considered by the mentioned researchers.

In the next section of the paper, the cash-in-advance model economy is described. Its state space, equilibrium conditions, and functional equations for the policy functions are presented and discussed. Section 3 details the implementation of the finite element parameterized expectations algorithm. Section 4 contains the benchmark calibration of the economy and a brief note on velocity, and Section 5 concludes.

2 Cash-in-Advance Model Economy

2.1 Discussion of the economy

In an economy with no price rigidities, a benevolent social planner maximizes the lifetime utility function of an infinitely lived representative agent by making choices over consumption, labor supply, next period capital, real holdings of a nominal bond, and real money holdings. Following the framework developed in Cooley and Hansen (1989), the consumption good is subject to a cash-in-advance constraint (CIA constraint, hereafter), and leisure and capital investment are considered to be credit goods. The agent does not draw utility directly from the real money holdings, money enters the framework for transaction purposes exclusively. The timing digression is as follows, the agent begins period t with money M_{t-1} and nominal bond holdings Z_{t-1}, and receives the lump-sum real transfer T_t. Prices are subsequently determined and the goods market opens.

Given k_0, the social planner chooses infinite sequences of consumption $\{c_t\}_{t=0}^{\infty}$, labor supply $\{n_t\}_{t=0}^{\infty}$, real holdings of a nominal bond $\{Z_t/p_t\}_{t=0}^{\infty}$, and real money balances $\{M_t/p_t\}_{t=0}^{\infty}$ for an infinitely lived agent, and an infinite sequence of capital stock $\{k_{t+1}\}_{t=0}^{\infty}$ to solve:

$$\max_{\left\{c_t,n_t,k_{t+1},\frac{M_t}{p_t},\frac{Z_t}{p_t}\right\}} E_t\left\{\sum_{t=0}^{\infty}\beta^t\left\{\frac{c_t^{1-\tau}-1}{1-\tau}-\gamma_n\frac{n_t^{1+\gamma}}{1+\gamma}\right\}\right\} \quad \beta\in(0,1) \tag{1}$$

subject to the budget constraint

$$c_t+k_{t+1}+\frac{Z_t}{p_t}+\frac{M_t}{p_t}=Y_t+(1-\delta)k_t+I_{t-1}\frac{Z_{t-1}}{p_t}+\frac{M_{t-1}}{p_t}+T_t \tag{2}$$

and a cash-in-advance constraint:

$$c_t\leq\frac{M_{t-1}}{p_t}+T_t \tag{3}$$

where $Y_t = Ak_t^{\alpha}n_t^{1-\alpha}$, $0<\alpha<1$, A is a production technology parameter, and $\delta\in[0,1]$ is the capital depreciation rate. Investment is defined as the next period capital stock minus the current period's undepreciated level of capital, i.e. $i_t = k_{t+1}-(1-\delta)k_t$. The holdings Z_{t-1} of a nominal bond yield the gross nominal return I_{t-1} from time $t-1$ to t

Changes in the money supply are realized through a lump-sum real transfer T_t to the agent, i.e. $T_t = (M_t - M_{t-1})/p_t$. The growth rate of the money supply θ_t, i.e. $\theta_t = M_t/M_{t-1}$, evolves according to a Markovian process with transitional probabilities Π. For the Q states of nature $\sum_{s=1}^{Q}\Pi_{ws}=1$. Typical element Π_{wr} is the probability of being on state r on time $t+1$ given the realization of state w in time t:

$$\Pi_{wr}=\Pr\left[\theta_{t+1}=\theta(r)|\theta_t=\theta(w)\right], \text{for } w \ \& \ r=1,...,Q \tag{4}$$

The budget constraint in eq. (2) implies that current period's consumption, real holdings of the nominal bond, real money holdings and next period capital stock are financed by actual production, undepreciated capital stock, returns on previously held bonds and the real money balances held from the previous period plus a lump-sum transfer. The CIA constraint requires the agent to hold money in order to purchase the consumption good.

Defining λ_t as the Lagrangian multiplier associated with the budget constraint and μ_t as the multiplier for the CIA constraint, the planner's problem is identified by first order conditions (5) – (9):

$$\lambda_t = c_t^{-\tau} - \mu_t \tag{5}$$

$$\gamma_n n_t^{\gamma} = \lambda_t (1-\alpha) \frac{Y_t}{n_t} \tag{6}$$

$$\lambda_t = \beta E_t \left\{ \lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{k_{t+1}} + (1-\delta) \right) \right\} \tag{7}$$

$$\frac{\lambda_t}{p_t} = \beta E_t \left\{ I_t \frac{\lambda_{t+1}}{p_{t+1}} \right\} \tag{8}$$

$$\frac{\lambda_t}{p_t} = \beta E_t \left\{ \frac{c_{t+1}^{-\tau}}{p_{t+1}} \right\} \tag{9}$$

the Kuhn-Tucker condition (10):

$$c_t \leq \left(\frac{M_{t-1}}{p_t} + T_t \right) \quad \text{and} \quad \mu_t \left[c_t - \left(\frac{M_{t-1}}{p_t} + T_t \right) \right] = 0 \tag{10}$$

and market clearing conditions for the goods market, the money market, and the nominal bond market, respectively (11) – (13):

$$c_t + k_{t+1} = Y_t + (1-\delta) k_t \tag{11}$$

$$M_t = M_{t-1} + p_t T_t \tag{12}$$

$$Z_t = Z_{t-1} = 0 \tag{13}$$

Eq. (5) shows the marginal benefit of consumption to be equal to the marginal utility of wealth plus the value of liquidity services needed to finance the transaction. A binding CIA constraint works as a necessary transaction cost which increases the marginal benefit of consumption at the equilibrium allocation. Eq. (6) is the condition for labor market equilibrium, where the marginal cost of labor supplied equals to the utility value of its marginal productivity. Eq. (7) is the Euler equation, portraying the relationship between current and expected future consumption decisions; if wealth was to be reduced slightly at the current period and carried over to the next period, the loss of current utility must equal the future discounted value of its real gross return. Equilibrium conditions (6) & (7) show how a binding CIA constraint creates a distortion away from consumption (cash goods) towards leisure and capital investment (credit goods). By raising the price of consumption above its production cost, the CIA constraint acts as a tax on consumption, diverting the planner towards acquiring goods which are not subject to this constraint. Eq. (9) shows how the current value of holding a unit of money in terms of utility, i.e. λ_t/p_t, equals the expected real marginal benefit of a unit of consumption next period. The Kuhn-Tucker condition (10) tells that if the constraint is binding, the slackness multiplier is positive, $\mu_t > 0$, and consumption must equal real money holdings, $c_t = M_t/p_t$; otherwise, if the constraint does not bind, the slackness multiplier is zero, $\mu_t = 0$, and real money holdings may be held for next period, $c_t \leq M_t/p_t$, in the form of precautionary demand for money. Table 1 indicates the formulae for calculating the consumption velocity of money, inflation, nominal interest, and real interest at time period t.

Insert Table 1 here

2.2 Steady states of the economy

Steady states of the discussed cash-in-advance economy are defined by equations (14) – (18):

$$\frac{k^{ss}}{y^{ss}} = \frac{\alpha\beta}{1-\beta(1-\delta)} \tag{14}$$

$$\frac{i^{ss}}{y^{ss}} = \delta\frac{k^{ss}}{y^{ss}} \tag{15}$$

$$\frac{c^{ss}}{y^{ss}} = 1 - \frac{i^{ss}}{y^{ss}} \tag{16}$$

$$n^{ss} = \left[\frac{\beta}{\theta^{ss}}\frac{(1-\alpha)}{\gamma_n}\left(\frac{c^{ss}}{k^{ss}}\right)^{-\tau}\left(\frac{y^{ss}}{k^{ss}}\right)^{\frac{\tau-\alpha}{1-\alpha}}\right]^{\frac{1}{\gamma+\tau}} \tag{17}$$

$$\mu^{ss} = \lambda^{ss}\left(\frac{\theta^{ss}}{\beta} - 1\right) \tag{18}$$

The capital stock to output ratio is derived from the Euler equation. The investment to output level is derived directly from the definition of investment and eq. (14). The consumption to output ratio is obtained from combining the goods market clearing condition with eq. (15). The steady state value of labor supply is derived using its first order condition, eq. (6), and the equilibrium condition for the bond holdings, eq. (8). The steady state of the CIA constraint is derived from the equilibrium condition for the holdings of the nominal bond and $I = \pi/\beta$. Notice the constraint is binding in the steady state for a steady state growth of money supply greater than the time discount factor.

2.3 Equilibrium of the economy

The equilibrium for the cash-in-advance economy is denoted by the sequence of real variables $\{\digamma_t\}_{t=0}^{\infty} = \{c_t, n_t, k_{t+1}, Z_t/p_t, M_t/p_t\}_{t=0}^{\infty}$, given a sequence of money stock growth rates $\{\theta_t\}_{t=0}^{\infty}$ evolving according to the transition matrix Π_{wr} in (4), a sequence of money supply and real lump-sum transfers $\{M_t, T_t\}_{t=0}^{\infty}$, and an initial stock of capital k_0, which satisfy the first order conditions (5), (6), (7), (8) & (9), the CIA constraint (10), and the market clearing conditions (11), (12) & (13).

2.4 State space & functional forms of the economy

The solution methodology employs the time invariant policy functions Υ_w & P_w, for all $w = [1, ..., Q]$, to express the equilibrium conditions of the cash-in-advance model economy. The partial state space Ω_t is composed of the possible realizations of the capital stock at time t and the money supply at time $t-1$. Conditional on the current realization of the money growth parameter $\theta_t = \theta(w)$, $\Upsilon_w(\Omega_t)$ is defined to map the current state of capital and previous state of money supply into the control of the conditional expectation function of the Euler equation, i.e. $\Upsilon_w(\Omega_t) \equiv \Upsilon_{t|\theta_t=\theta(w)} = \Upsilon(k_t, M_{t-1}|\theta_t = \theta(w))$ where $\Upsilon_{t|\theta_t=\theta(w)} = \beta E_t\{\lambda_{t+1}\alpha\frac{Y_{t+1}}{k_{w,t+1}} + (1-\delta)\}$. Also conditional on the realization of money growth, $P_w(\Omega_t)$ is defined to map the current state of capital and previous money supply into the control of the price function, i.e. $P_w(\Omega_t) \equiv p_{t|\theta_t=\theta(w)} = P(k_t, M_{t-1}|\theta_t = \theta(w))$. The capital domain is bounded above by a maximum sustainable capital stock $\bar{\mathrm{k}}$, and bounded below by $\underline{\mathrm{k}}$. The money stock domain is also bounded above and below. Ω has a well defined compact support, i.e. $\Omega = [\underline{\mathrm{k}}, \bar{\mathrm{k}}] \times [\underline{\mathrm{M}}, \bar{\mathrm{M}}]$.

Define $\vec{\Upsilon}$ and $\vec{P}$ as column vectors containing the policy functions Υ_w and P_w for all $w \in \{1, ..., Q\}$, i.e. $\vec{\Upsilon} = [\Upsilon_1, ..., \Upsilon_Q]'$ $\vec{P} = [P_1, ..., P_Q]'$. Given the policy functions $\Upsilon_w(k_t, M_{t-1})$ and $P_w(k_t, M_{t-1})$, for all w, the residuals of the Euler equation and the equilibrium condition for real money holdings can be written as in eqs. (19) & (20), respectively:

$$R_w^K\left(k, M_{-1}; \vec{\Upsilon}, \vec{P}\right) = \Upsilon_w(k, M_{-1}) - \beta \sum_{z=1}^{Q} \Pi_{wz}\{\Upsilon_z(\tilde{k}_w, M)[\alpha A \tilde{k}_w^{\alpha-1} \tilde{n}_z^{1-\alpha} + (1-\delta)]\} \tag{19}$$

$$R_w^M\left(k, M_{-1}; \vec{\Upsilon}, \vec{P}\right) = \frac{\Upsilon_w(k, M_{-1})}{P_w(k, M_{-1})} - \beta \sum_{z=1}^{Q} \Pi_{wz}\left\{\frac{\tilde{c}_z^{-\tau}}{P_z(\tilde{k}_w, M)}\right\} \tag{20}$$

for all $w = \{1, ..., Q\}$. Where the real variables are defined by

$$\tilde{k}_w = A k^\alpha n_w^{1-\alpha} + (1-\delta) k - c_w \tag{21}$$

$$n_w = \left[\frac{(1-\alpha)}{\gamma_n} \Upsilon_w(k, M_{-1}) A k^\alpha\right]^{\frac{1-\alpha}{\gamma+1}} \tag{22}$$

$$c_w = \begin{cases} \Upsilon_w(k, M_{-1})^{-1}, & if\ \mu_w = 0 \\ M/P_w(k, M_{-1}), & if\ \mu_w > 0 \end{cases} \tag{23}$$

and next period's money supply is

$$M = \theta(w) M_{-1} \tag{24}$$

The policy functions in $\vec{\Upsilon}$ and $\vec{P}$ are the solutions to $R_w^K(k, M_{-1}; \vec{\Upsilon}, \vec{P}) = 0$ and $R_w^M(k, M_{-1}; \vec{\Upsilon}, \vec{P}) = 0$, for all w, and in combination with $\tilde{k}_w$, n_w, c_w & M from eqs. (21), (22), (23) & (24) and the sequence of money growth rates θ, evolving according the transition matrix Π in (4), they generate the sequences $\{c_t$, n_t, $k_{t+1}, Z_t/p_{t-1}$, $M_t/p_{t-1}\}_{t=0}^{\infty}$, that solve for the equilibrium of this economy along the space Ω, for each $\theta(w)$.

3 Solution Methodology

The solution procedure involves a parameterized expectations algorithm to minimize the weighted absolute value of residual functions $R_w^K(k, M_{-1}; \vec{\Upsilon}, \vec{P})$ and $R_w^M(k, M_{-1}; \vec{\Upsilon}, \vec{P})$; for all w, where the true decision rules $\Upsilon_w(k, M_{-1})$ & $P_w(k, M_{-1})$ are replaced by the parametric approximations $\upsilon_w^h(k, M_{-1})$ & $p_w^h(k, M_{-1})$, for $w = \{1, ..., Q\}$. υ_w^h & p_w^h are approximated using an implementation of the finite element method, that closely follows that advocated in McGrattan (1996).

To create the approximate time invariant functions υ_w^h & p_w^h, the space $\Omega = [\underline{\mathrm{k}}, \bar{\mathrm{k}}] \times [\underline{\mathrm{M}}, \bar{\mathrm{M}}]$ is divided in n_e nonoverlapping rectangular subdomains called "elements". Parameterizations of the functions for each element, at each realization of w, are constructed using linear combinations of low order polynomials or "basis functions"; creating local approximations for each function.

The parametrized functions $\upsilon_w^h(k, M_{-1})$ & $p_w^h(k, M_{-1})$ are built as follows:

$$\upsilon_w^h(k, M_{-1}) = \sum_{ij}^{IJ} \upsilon_{ij}^w W_{ij}(k, M_{-1}) \tag{25}$$

$$p_w^h(k, M_{-1}) = \sum_{ij}^{IJ} p_{ij}^w W_{ij}(k, M_{-1}) \tag{26}$$

where $i = \{1, ..., I\}$ indicate capital nodes, $j = \{1, ..., J\}$ indicate money supply nodes, $W_{ij}(k, M_{-1})$ is a set of linear basis functions dependant on the element $[k_i, k_{i+1}] \times [M_j, M_{j+1}]$, for all i, j, over which the local approximations are performed. υ_{ij}^w & p_{ij}^w are vectors of coefficients to be determined. The parameterized value of the conditional expectation function and the price function, over the complete space Ω and for each w, are obtained by piecing together all the local approximations. The approximate solutions for $\upsilon_w^h(k, M_{-1})$ & $p_w^h(k, M_{-1})$ on Ω for each w are then "piecewise linear functions".

Finite element methods are often characterized by the shape of the basis functions $W_{ij}(k, M_{-1})$ they employ. These are constructed such that they take a value of zero for most of the space Ω, except for a small interval where they take a simple linear form. The basis functions adopted for these approximations are set such that $W_{ij}(k, M_{-1}) = \Psi_i(k)\,\Phi_j(M_{-1})$, where

$$\Psi_i(k) = \begin{cases} \frac{k-k_i}{k_{i+1}-k_i} & \text{if} \quad k \in [k_{i-1}, k_i] \\ \frac{k_{i+1}-k}{k_{i+1}-k_i} & \text{if} \quad k \in [k_i, k_{i+1}] \\ 0 & \text{elsewhere} \end{cases} \tag{27}$$

$$\Phi_j(M_{-1}) = \begin{cases} \frac{M_{-1}-M_j}{M_{j+1}-M_j} & \text{if} \quad M_{-1} \in [M_{j-1}, M_j] \\ \frac{M_{j+1}-M_{-1}}{M_{j+1}-M_j} & \text{if} \quad M_{-1} \in [M_j, M_{j+1}] \\ 0 & \text{elsewhere} \end{cases} \tag{28}$$

for all $i,\ j$. $\Psi_i(k)$ & $\Phi_j(M_{-1})$ have the shape of a continuous pyramid which peaks at nodal points $k = k_i$ & $M_{-1} = M_j$, respectively, and are only non-zero on the surrounding elements of these nodes.

The approximations $\upsilon_w^h(k, M_{-1})$ & $p_w^h(k, M_{-1})$, for all w, are chosen to simultaneously satisfy the equations:

$$\begin{aligned} \int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} \omega(k, M_{-1})\, R_w^K\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM &= 0, \\ \text{for } w &= \{1, ..., Q\} \end{aligned} \tag{29}$$

$$\begin{aligned} \int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} \omega(k, M_{-1})\, R_w^M\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM &= 0, \\ \text{for } w &= \{1, ..., Q\} \end{aligned} \tag{30}$$

where $\omega(k, M_{-1})$ is a weighting function, and $R_w^K(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h)$ & $R_w^M(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h)$ are the residuals from the Euler equation and the equilibrium condition for real money holdings, where the true policy functions $\vec{\Upsilon}$ & $\vec{P}$ are replaced by the vectors of parametric approximations $\vec{\upsilon}^h$ & $\vec{p}^h$. A Galerkin scheme is employed to find the vectors of coefficients υ_{ij}^w & p_{ij}^w, for all i, j and all w, which solves for the weighted residual equations (29) & (30) over the complete space Ω and each w. The Galerkin scheme employs the basis functions $W_{ij}(k, M_{-1})$ as weights for the residual equations $R_w^K(k, M_{-1}; \vec{\upsilon}^h, \vec{p}^h)$ & $R_w^M(k, M_{-1}; \vec{\upsilon}^h, \vec{p}^h)$:

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} W_{ij}(k, M_{-1}) R_w^K\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM = 0 \ \ \forall \ \ i, \ j \ \ \& \text{ all } w \tag{31}$$

$$\int_{\underline{\mathrm{M}}}^{\bar{\mathrm{M}}} \int_{\underline{\mathrm{k}}}^{\bar{\mathrm{k}}} W_{ij}(k, M_{-1}) R_w^M\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM = 0 \ \ \forall \ \ i, \ j \ \ \& \text{ all } w \tag{32}$$

Since the basis functions are only nonzero surrounding their nodes, eqs. (31) & (32) can be rewritten in terms of the individual elements:

$$\sum_{e=1}^{n_e} \int_{\Omega_e} W_{ij}(k, M_{-1}) R_w^K\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM = 0 \ \ \forall \ \ i, \ j \ \ \& \text{ all } w \tag{33}$$

$$\sum_{e=1}^{n_e} \int_{\Omega_e} W_{ij}(k, M_{-1}) R_w^M\left(k, M_{-1};\ \vec{\upsilon}^h, \vec{p}^h\right) dk dM = 0 \ \ \forall \ \ i, \ j \ \ \& \text{ all } w \tag{34}$$

where n_e is the total number of elements and Ω_e is the capital and money stock domain covered by the element e.

A Newton algorithm is used to find the coefficients for $[\vec{\upsilon}_s, \vec{p}_s]$ which solve for the nonlinear system of equations H:

$$H([\vec{\upsilon}_s, \vec{p}_s]) = 0 \tag{35}$$

where $H([\vec{\upsilon}_s, \vec{p}_s])$ is denoted by eqs. (33) & (34). The first step is to choose initial vectors of coefficients $[\vec{\upsilon}_{s_0}, \vec{p}_{s_0}]$, and iterate as follows:

$$\left[\vec{\upsilon}_{s_{l+1}}, \vec{p}_{s_{l+1}}\right] = [\vec{\upsilon}_{s_l}, \vec{p}_{s_l}] - J([\vec{\upsilon}_{s_l}, \vec{p}_{s_l}])^{-1} H([\vec{\upsilon}_{s_l}, \vec{p}_{u_l}]) \tag{36}$$

where J is the Jacobian matrix of H, and $[\vec{v}_{s_l}, \vec{p}_{s_l}]$ is the l^{th} iteration of $[\vec{v}_s, \vec{p}_s]$. The algorithm solves for the parameterized version of the conditional expectation function of the Euler equation $v_w^h(k, M_{-1})$ and for the price function $p_w^h(k, M_{-1})$, for each possible w and all $\{k, M_{-1}\} \subset \Omega$. Convergence is assumed to have occurred once $\left\| \left[\vec{v}_{s_{l+1}}, \vec{p}_{s_{l+1}}\right] - [\vec{v}_{s_l}, \vec{p}_{s_l}] \right\| < 10^{-7}$. The Appendix to this paper describes the steps followed for the implementation of the algorithm.

4 Business Cycle Properties

4.1 Calibration

For the model calibration, the time interval is set to a quarter of a year, the time discount factor is set to 0.99, depreciation rate at 0.019, the capital elasticity of output to 1/3, the inverse of labor supply elasticity is set to 0, denoting Hansen (1985) indivisible labor, steady state labor supply to 0.31, and steady state inflation is set to the U.S. quarterly data mean value from 1959:III to 1998:III, that is 1.727%. This calibration of parameters is comparable to that of Walsh (1998). For the benchmark calibration, following Kocherlakota (1996) and King and Rebelo (1999), the intertemporal elasticity of substitution in consumption (IES, hereafter) is set to equal 1/3, yielding $\tau = 3.00$. Other values for IES are also considered on the analysis.

The growth rate of the money supply θ_t is assumed to evolve according to the AR(1) process

$$\theta_{t+1} = (1-\rho)\theta^{ss} + \rho\theta_t + \varepsilon_{t+1} \quad where\ \varepsilon\tilde{\ }N\left(0, \sigma_\varepsilon^2\right)$$

The result of a first order autoregressive estimation of M2, using quarterly data from 1959:III to 1998:III, yields $\rho = 0.727$ and $\sigma_{\varepsilon} = 0.01$. The AR(1) process is approximated by a discrete Markov chain using the methodology advanced by Tauchen and Hussey (1991). Also, given the high persistence of the process, the adjustment to the weighting function recommended in Flodén (2008) is performed. Five states for the money supply growth rate are considered; these return the state vector $\theta = [0.9846, 1.0017, 1.0173, 1.0328, 1.0500]$, and the state transition matrix in (37). Table 2 summarizes the parameters used on the benchmark calibration of the economy.

$$\Pi = \begin{bmatrix} 0.51136 & 0.45235 & 0.03601 & 0.00028 & 0.00000 \\ 0.07677 & 0.58264 & 0.32302 & 0.01751 & 0.00005 \\ 0.00361 & 0.19100 & 0.61076 & 0.19100 & 0.00361 \\ 0.00005 & 0.01751 & 0.32302 & 0.58264 & 0.07677 \\ 0.00000 & 0.00028 & 0.03601 & 0.45235 & 0.51136 \end{bmatrix} \tag{37}$$

Insert Table 2 here

The procedure employed in finding the steady state values of the endogenous variables of the model requires the calculation of $\upsilon_w^h(k, M_{-1})$ and $p_w^h(k, M_{-1})$ for a determinate environment, and finding k, such that $k = \tilde{k}_w$ and that the steady state ratios contained in equations (14) & (18) are realized. This is done by minimizing the geometric distance between the vector $\vec{x}^{ss}$, containing the values of the analytical steady state and the vector $\vec{x}_w^h$, containing the steady state approximations, where $\theta(w) = \theta^{ss}$. Vectors $\vec{x}^{ss}$ & $\vec{x}_w^h$ are defined in equations (38) and (39), respectively, and the geometric distance is defined in equation (40). Table 3 presents the steady state values for the endogenous variables of the economy at the benchmark calibration.

$$\vec{x}^{ss} = \left[k^{ss}, \frac{k^{ss}}{y^{ss}}, \frac{i^{ss}}{y^{ss}}, \frac{c^{ss}}{y^{ss}}, n^{ss}, \mu^{ss}\right]' \tag{38}$$

$$\vec{x}^{h}_{w} = \left[\tilde{k}_w, \frac{k}{Ak^{\alpha}n_w^{1-\alpha}}, \frac{\tilde{k}_w - (1-\delta)\,k}{Ak^{\alpha}n_w^{1-\alpha}}, \frac{c_w}{Ak^{\alpha}n_w^{1-\alpha}}, n_w, c_w^{-\tau} - \Upsilon_w(k, M_{-1})\right]' \tag{39}$$

$$\left\|\vec{x}^{ss} - \vec{x}^{h}_{w}\right\| = \sqrt{\sum_{i=1}^{6}\left(x_i^{ss} - x_i^{h}\right)^2} \tag{40}$$

Insert Table 3 here

Figure 1 presents a mesh illustrating the level values of the CIA constraint for the state space $\Omega = \left[\underline{\mathrm{k}}, \bar{\mathrm{k}}\right] \times \left[\underline{\mathrm{M}}, \bar{\mathrm{M}}\right]$, and all the realizations w of the monetary growth parameter, and a two dimensional graph representing the values of the CIA constraint for a single value of M_{-1}. The CIA constraint is always binding at the steady state level of capital stock, except at the lowest realization of the money growth parameter. Due to the serial correlation of monetary growth, at current low realizations of the shock, the agent chooses to carry real money holdings for the next period due to the fear of future low realizations of the shock and the need to smooth future consumption with current. As a consequence, money is held for precautionary purposes. From Figure 1 is also possible to notice that, even at the low realizations of the growth parameter, the agent may choose not to carry money for next period at low accumulations of capital. Below a critical level of capital stock, the real interest is sufficiently high to dissuade the agent from holding any money for precautionary reasons, and instead acquire interest bearing physical capital.

Insert Figure 1 here

4.2 Note on Variability of Velocity

This note serves to shed light on the variability of velocity, due to serially correlated shocks to the money growth rate, on a modeling environment that has become a workhorse of monetary optimal growth literature. The ability of the methodology to approximate the solution of the model, while allowing for the CIA constraint to occasionally bind, permits to perform an analysis on the variability of velocity of money. Velocity varies when the real money holdings by an agent on a period are greater than the consumption expenditures, e.g. $V = PC/M$, and will stay constant at one when the choice of investing in an interest bearing venture, such as capital, is preferred to holding additional units of cash. The latter occurs when real interest rate is sufficiently high and the variation of expected marginal utility of consumption is relatively small, Hodrick et al. (1991). On the model economy, variable velocity occur when individuals accumulate money holdings for precautionary purposes from a period to the next, and it is triggered on the realizations of low money growth rates on a period and hold on to cash for next period to smooth their consumption.

Table 4 presents the simulated moments of the model at different levels of the intertemporal elasticity of substitution (or IES) and the sample moments for the US economy for the time period 1959:II–1988:I.[1] The variability of velocity is measured by its coefficient of variation, i.e. $cv(V) = (\sigma_V/\mu_V) \cdot 100$. As in Hodrick et al. (1991), the existence of credit goods, alongside cash goods in the economy, generates a variability of velocity without the need of frictions on the information structure, as proposed by Svensson (1985). Still, the model is not able to generate

[1] Period covered by the research on velocity of Hodrick et. al. (1991) and Shi (2006).

realistic first and second moments for velocity. At the benchmark calibration, shocks to the rate of money growth are able to explain less than two percent of the total variability in velocity observed on the period, and this percentage drops as the value of IES increases. Because of these results, reseachers intending to reconcile the velocity predictions of a realistically calibrated cash-in-advance model economy with those observed in the data tend to depart and modify the Cooley and Hansen (1989) formulation of the economic environment; i.e, Svensson (1985), Hodrick et. al. (1991), Hromcová (2008) and Shi (2006). The proposed solution methodology in this paper could be fitted to solve for comparable modeling environments to the ones studied by the mentioned researchers and improve the fit of velocity variability predictions, without relying on unrealistic values of IES or excessive variability of the money growth parameters. Nevertheless, the model as is, under a realistic parameterization of monetary shocks, yields sample comovements of selected nominal and real variables that, with respect to direction and somewhat to magnitude, are consistent with those of the data. Although not presented in Table 4, these results hold for different calibrations of the parameter denoting labor supply elasticity.

Table 4 also shows selected moments for a model calibration representing logarithmic preferences on the agent's consumption decisions, i.e. $\tau = 1$, regularly used on the empirical literature. For none of the simulation trials, velocity was distinct from one on the considered state space. This result is highly significant because it implies that a solution methodology assuming an always binding CIA constraint can be considered appropriate for such calibration of the model economy; as it is assumed by, for example, most implementations of a logarithmic linearization of the first order conditions of the model economy.

Insert Table 4 here

5 Conclusion

This paper proposes a finite element approach and a parameterized expectations algorithm to solve for the equilibrium of a cash-in-advance constraint model economy. While the solution procedure benefits from the high accuracy and stability of finite elements, by performing a parameterization of the expectation component of the policy functions, it permits the CIA constraint to occasionally bind allowing an analysis on velocity; for which a note is included in the paper. For a practitioner interested in fitting the predictions of a cash-in-advance model economy to those of the data, the proposed methodology can be extended to incorporate most departures of the studied economic setting that are currently studied by the academic literature.

References

1. Aruoba, S.B., Rubio, J., Fernandez-Villaverde, J., 1998. Comparing Solution Methods for Dynamic Equilibrium Economies. Journal of Economic Dynamics and Control 30, 2477-2508

2. Christiano, L., Fisher, J., 2000. Algorithms for solving dynamic models with occasionally binding constraints. Journal of Economic Dynamics and Control 24, 1179-1232.

3. Clower, R.W., 1965. A Reconsideration of the Microfoundations of Monetary Theory. Western Economic Journal 6, 1-9.

4. Collard, F., Fève, P., Parraudine, C., 2000. Solving and Estimating Dynamic Models under Rational Expectations. Computational Economics 15, 201-221.

5. Cooley, T.F., Hansen, G., 1989. The inflation tax in a real business cycle model. American Economic Review 79, 733-48.

6. Floden, M., 2008. A Note on the Accuracy of Markov-Chain Approximations to Highly Persistent AR(1)-Processes. Economic Letters 99, 516-520.

7. Hansen, G., 1985, Indivisible labor and the business cycle. Journal of Monetary Economics 16, 309-325.

8. Hodrick, R., Kocherlakota, N., Lucas, D., 1991. The Variability of Velocity in Cash-in-Advance Models. The Journal of Political Economy 99, 358-384.

9. Hromcová, J., 2008. Learning-or-doing in a cash-in-advance economy with costly credit. Journal of Economic Dynamics and Control 32, 2826-2853.

10. Judd, K.L., 1998, Numerical Methods in Economics. The MIT Press, Cambridge, Massachussetts.

11. King, R., Rebelo, S., 1999. Resuscitating Real Business Cycles, in: Taylor, J. B., Woodford, M. (Eds.), Handbook of Macroeconomics, North-Holland, Amsterdam, pp. 927-1007.

12. Kocherlakota, N., 1996. The Equity Premium: It's Still a Puzzle. Journal of Economic Literature 34, 42-71.

13. Lucas Jr., R.E., 1982. Interest Rates and Currency Prices in a Two Country Model. Journal of Monetary Economics 10, 335-59.

14. Lucas Jr., R.E., Stockey, N., 1987. Interest in a Cash in Advance Economy. Econometrica 55, 491-513.

15. McGrattan, E., 1996. Solving the Stochastic Growth Model With a Finite Element Method. Journal of Economic Dynamics and Control 20, 19-42.

16. McGrattan, E., 1999. Application of Weighted Residual Methods to Dynamic Economic Models, in: Marimon, R., Scott, A. (Eds.), Computational Methods for the Study of Dynamic Economies, Oxford University Press.

17. Svensson, L., 1985. Money and Asset Prices in a Cash in Advance Economy. Journal of Political Economy 93, 919-944.

18. Tauchen, G, Hussey, R., 1991. Quadrature-based methods for obtaining approximate solutions to nonlinear pricing models. Econometrica 59, 371-396.

19. Walsh, C.E., 1998. Monetary Theory and Policy. Second Edition. The MIT Press, Cambridge, Massachussetts.

Appendix

Algorithm Procedure

The chronology of the algorithm employed in solving the model can be described in the following steps:

1. Choose the location and quantity I & J of nodes along the capital and money supply domain (i.e. k_i & M_j for $i = \{1, ..., I\}, \;\; j = \{1, ..., J\}$), which will delimit the n_e nonoverlapping elements in the state space Ω, and use a Gaussian-Legendre quadrature rule to identify abscissas and weights of capital and technology on each element.

2. Identify the different states $u(w)$ & $u(r)$, for $w, r = 1, ..., Q$, of the Markovian monetary shock, and the transition matrix Π_{wr} of the process.

3. For realization w of the monetary shock, use the parameterization $\upsilon_w^h\left(k, M_{-1}\right)$ to approximate the value of Υ_w, following eq. (25), and $p_w^h\left(k, M_{-1}\right)$ to approximate the value of p_w, following eq. (26), for all Gauss-Legendre capital and money stock abscissas of each element along the state space Ω.

4. Initiate a recursive solution procedure by creating the conjecture that the cash-in-advance constraint for this economy, eq. (10), does not bind, i.e. $\mu_w = 0$, & $c_w \leq M/p_w$. This implies $c_w^{-\tau} = \lambda_w \equiv \upsilon_w^h\left(k, M_{-1}\right)$. Consumption is automatically solved, i.e. $c_w = \lambda_w^{-\tau}$.

5. Compute the values of n_w, using eq. (22), of actual output Y_w, and of real money holdings $M/p_w^h\left(k, M_{-1}\right)$.

6. Check whether the initial conjecture was correct. If "yes", go to next step.

   If not, the constraint binds: $\mu_w > 0$ & $c_w = M/p_w^h(k, M_{-1})$. The value of the CIA multiplier becomes $\mu_w = [c_w^{-\tau} - \upsilon_w^h(k, M_{-1})]$.

7. Obtain next period's capital $\tilde{k}_w$ using (21), and calculate investment, and next period's stock of money, using (24), for all possible realizations of shock $\theta(z)$.

8. Use the parameterizations $\Upsilon_z(\tilde{k}, M)$ & $P_z(\tilde{k}, M)$ to approximate the values of Υ_z & p_z for the choice of k_{t+1} and all the z possible future of realizations of θ_{t+1}

9. For each z, create a conjecture that the cash-in-advance constraint does not bind at time $t+1$, i.e. $\mu_z = 0,$ & $c_z \leq \tilde{M}/p_{z,t+1},$ implying $\tilde{c}_z^{-\tau} = \tilde{\lambda}_z \equiv \upsilon_z^h(\tilde{k}, M)$. Consumption is automatically solved, i.e. $\tilde{c}_z = \tilde{\lambda}_z^{-\tau}$.

10. Compute the value of each possible $\tilde{n}_z$, $\tilde{Y}_{z.}$, and $\tilde{M}/p_z^h(\tilde{k}, M)$.

11. Check whether the conjecture in Step 9 was correct. If "yes", go to next step.

    If not, the constraint binds: $\tilde{\mu}_z > 0,$ & $\tilde{c}_z = \tilde{M}/p_z^h(\tilde{k}, M)$. The value of the CIA multiplier becomes $\tilde{\mu}_z = [\tilde{c}_z^{-\tau} - \upsilon_z^h(\tilde{k}, M)]$.

12. Construct the residual functions $R_K\left(k, M; \vec{\upsilon}^h, \vec{p}^h\right)$ & $R_M\left(k, M; \vec{\upsilon}^h, \vec{p}^h\right)$ using eqs. (19) and (20) for each state w.

13. If the weighted approximations of the residual functions for each element, as in eq. (33) and (34), are sufficiently close to zero then "stop", else update

the vectors of coefficients υ_{ij}^{w} & p_{ij}^{w}, for all w, as in (36) and go to Step 3.

Tables and Figures

Table 1: Expressions of some endogenous variables

Consumption Velocity	$V_t = \frac{C_t}{M_t/p_t}$
Inflation	$\pi_t = \frac{p_t}{p_{t-1}}$
Nominal Interest	$I_t = E_t\left\{c_{t+1}^{-\tau}\frac{1}{\pi_{t+1}}\right\}/E_t\left\{\lambda_{t+1}\frac{1}{\pi_{t+1}}\right\}$
Real Interest	$r_t = E_t\left\{\lambda_{t+1}\left(\alpha\frac{Y_{t+1}}{k_{t+1}} + (1-\delta)\right)\right\}$

Table 2: Benchmark Economy

Parameters								
A	α	β	δ	γ	τ	θ^{ss}	ρ	σ_ε
1.00	0.33	0.99	0.02	0.00	3.00	1.01727	0.73	0.01

Table 3: Benchmark Economy

Steady States								
c^{ss}	k^{ss}	n^{ss}	Y^{ss}	μ^{ss}	π^{ss}	r^{ss}	I^{ss}	V^{ss}
0.82	11.94	0.31	1.04	0.05	1.017	1.01	1.03	1.00

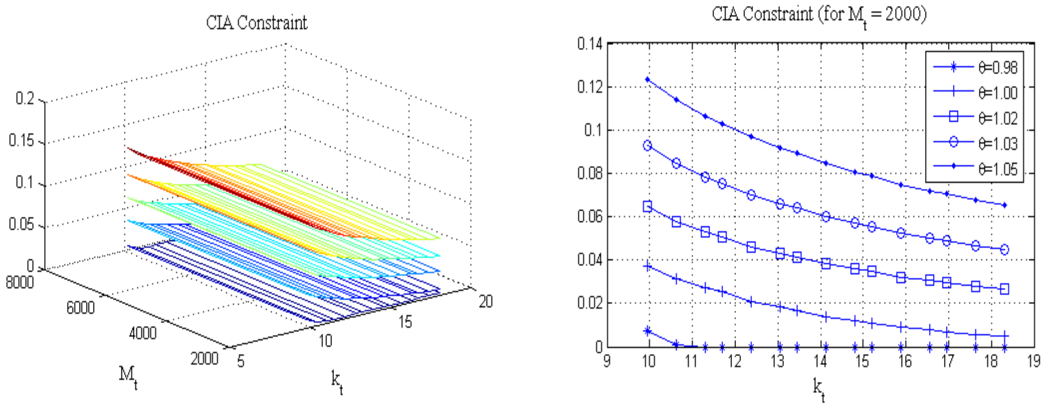

Figure 1: Cash-in-Advance Constraint

Table 4: Simulated Moments

	$\tau = 1.0$	$\tau = 2.5$	$\boldsymbol{\tau = 3.0}$	$\tau = 3.5$	Data
$cv\,(V)$	0	0.0210	**0.0306**	0.0385	1.9719[2]
	n.a.	(0.0081)	**(0.0127)**	(0.0161)	
$corr\left(V, \frac{Y_t}{Y_{t-1}}\right)$	0	-0.1844	**-0.1788**	-0.1734	-0.3420[3]
	n.a.	(0.0710)	**(0.0728)**	(0.0714)	
$corr\left(V, \frac{C_t}{C_{t-1}}\right)$	0	-0.1634	**-0.1534**	-0.1442	-0.3537[3]
	n.a.	(0.0672)	**(0.0663)**	(0.0637)	
$corr\,(V, I)$	0	0.4319	**0.4243**	0.4167	0.5165[3]
	n.a.	(0.1472)	**(0.1535)**	(0.1540)	
$corr\,(\pi, I)$	0.7059	0.7277	**0.7300**	0.7309	0.5135[3]
	(0.0334)	(0.0399)	**(0.0413)**	(0.0407)	
$corr\,(\pi, r)$	-0.5166	-0.5691	**-0.5768**	-0.5839	-0.4940[3]
	(0.0326)	(0.0375)	**(0.0371)**	(0.0364)	
$corr\,(\pi, \theta)$	0.1506	0.2535	**0.2626**	0.2816	0.1844[4]
	(0.1192)	(0.1264)	**(0.1226)**	(0.1287)	

Numbers in parenthesis are standard deviations over 500 simulations.

[2]Source: Shi (2006)

[3]Source: Hodrick et al. (1991).