Demand Uncertainty Leads to Diverse Collusive Dynamics

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Abstract

We characterize collusive behavior in a model where firms' must pre-commit to the scale of operation before observing market demand and setting prices. Firms are uncertain about future demand; each period they choose capacity when demand is uncertain, then observe demand and choose prices. In this model, collusive equilibria can have highly varied behavior on a single equilibrium path; collusion can involve asymmetric firm size (capacities), periods of monopoly pricing, symmetric pricing below the monopoly level, asymmetric pricing, and mixed strategy price wars.

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1 Introduction

We characterize collusive behavior in a model of industries where firms' must pre-commit to their scale of operation before observing market demand and setting prices. Firms are uncertain about future demand; each period they choose capacity when demand is uncertain, then observe demand and choose prices. Collusive equilibria can have highly varied behavior on a single equilibrium path; collusion can involve asymmetric firm size, periods of monopoly pricing, symmetric pricing below the monopoly level, asymmetric pricing, and mixed strategy price wars. In periods of extremely high demand, collusive prices are at constrained monopoly levels. In extremely low demand periods collusive pricing is symmetric and between monopoly pricing and marginal cost. In demand periods between these extremes, prices can be asymmetric with one firm accepting the residual demand and charging the monopoly price, while the other firm charges a price lower.

Our model can be viewed as advancing two well established literatures. The first is a large literature that studies collusive pricing patterns with non-stationary demand (This literature includes: Green and Porter (1983), Rotemberg and Saloner (1986), Kandori (1991), Haltiwanger and Harrington (1991), Bagwell and Staiger (1997)). The second literature deals with endogenous capacities and repeated price competition with stationary demand (This literature includes: Davidson and Deneckere (1984), Brock and Scheinkman (1985), Benoit and Krishna (1987), Lambson (1987,1994), Davidson and Deneckere (1990), Compte, Jenny and Rey (2002) and Dechenaux and Kovenock (2010)).

In particular, we follow a recent literature that integrates capacity constraints into dynamic games with non-stationary demand (Staiger and Wolak (1992), Fabra (2006) and Knittel and Lepore (2010)).[1] Our point of departure from this literature is Staiger and Wolak (1992), whose model includes repeated price competition with independently identically distributed demand shocks each period. Capacities are chosen before the demand state is realized while prices are chosen after. They restrict analysis to exclusively symmetric capacities, simplifying the analysis greatly.[2] They find that collusive pricing might involve periods of both mild and severe price wars. A *mild* price war is a joint lowering of symmetric prices below the monopoly level, while a *severe* price war is a period where firms employ noncooperative mixed pricing on the collusive path. Because the severe price wars can be on the equilibrium path, collusive market shares are unstable over time.

In our analysis, we *do not* restrict firms to have symmetric capacities and, to our knowledge, are the first to do so in a model with demand uncertainty. Two papers Compte, Jenny

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[1] Both Fabra (2006) and Knittel and Lepore (2010) extend the deterministic demand cycle model of Haltiwanger and Harrington (1991); Fabra adds exogenous symmetric capacity constraints, while Knittel and Lepore add endogenous long-run capacities which can be asymmetric.

[2] Staiger and Wolak (1992) incorrectly claim that pure symmetric capacities subgame perfect equilibrium always exist. If one reinterpretes their paper to only adress collusion with the best symmetric capacities, as we have done above, then their primary results are recovered.

and Rey (2002), and Dechenaux and Kovenock (2010) analyze collusion with asymmetric capacities, while assuming demand is constant over time.[3] Compte, Jenny and Rey allow for asymmetric capacities, but restrict their analysis to subgame perfect equilibria with pure symmetric pricing. The focus of their paper is on merger analysis, and they show that joint collusive profits are often increased with firm symmetry.[4] Dechenaux and Kovenock study the same basic model as Compte, Jenny and Rey, but instead characterize joint-profit maximizing collusion allowing for any pure prices. At many asymmetric capacities/discount pairs prices are asymmetric where the larger firm prices to maximize the residual demand, and the smaller firm prices as high as possible while keeping the larger firm from undercutting.

We consider concave demand functions and endogenous capacities that are free to be (and often are) asymmetric. In the context of the literature, our model can be viewed as a generalization of Staiger and Wolak (1992) to concave demand (they use a linear specification) in which we consider equilibria (similar to Dechenaux and Kovenock (2010)) with both asymmetric capacities and asymmetric pricing.[5],[6]

The characterization of the collusive pricing paths with asymmetric prices is diverse in content. We provide some sufficient conditions for the firms' to prefer to price at the monopoly level, at a symmetric price lower than monopoly, and asymmetrically in pure strategies; in a given demand realization on the equilibrium pricing path. When pricing is asymmetric it involves one firm charging a price at or below the monopoly price while the other firm charges a price lower, which is determined by the first firm's temptation to cheat. This is similar to the type of collusive 'judo pricing,' shown in Dechenaux and Kovenock (2010).[7]

The remainder of the paper is organized as follows. In section 2. we lay out the basic assumptions of our model, describe non-cooperative behavior, and the joint profit maximizing behavior of the firms. Section 3 deals with most-collusive pricing, we separately deal with symmetric and asymmetric case and then provide a synthetic characterization. A final subsection presents numerical examples for model. Section 4 considers the collusive capacities and we conclude with section 5.

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[3] In another closely related paper, Knittel and Lepore (2010) allow for asymmetric capacities, but focus on symmetric pricing in a model where demand fluctuates based on deterministic cycles over time.

[4] Both Compte, Jenny and Rey (2002) and Dechenaux and Kovenock (2010) assume rectangular shaped market demand. We consider concave demand functions; the rectangular demand is a limiting case of our model just outside assumptions.

[5] Staiger and Wolak (1992) use efficient rationing of residual deamnd, while we use the proportional rationing.

[6] Dechenaux and Kovenock (2010) assume for endogenous rationing of collusive demand. This assumption allows firms to collude in price and the rationing of collusive demand. In contrast, we assume that firms only collude in prices, but cannot collude in how demand is rationed.

[7] The concept of "judo pricing" was developed in Gelman and Salop (1983). They consider a model of entry where the entrant chooses capacity and price followed by and incumbent (with unlimited capacity) choosing price. The optimal strategy of the entrant is to choose capacity and price low enough that the incumbent is better off pricing as a monopolist and taking the residual demand than undercutting.

2 Model Basics

Consider an industry with two firms that produce a single homogeneous product. The index i is used to identify an arbitrary firm, where $i \in \{1, 2\}$. Throughout the paper we will use j to index the firm other than i. Demand fluctuates over time according to random shocks. Each period firms choose capacities before market demand is observed. Then demand is observed and firms choose their prices. We assume that demand is drawn independently from an identical distribution (*iid*) each period. Let $\omega_t \in \Omega$ be a demand state for time t, where $\Omega = [\underline{\omega}, \overline{\omega}] \subset \mathbb{R}$. The market demand function at time t, given the state $\omega_t \in \Omega$, is $D(\cdot, \omega_t) : \mathbb{R} \mapsto \mathbb{R}_+$. The inverse demand for any time t, given the state $\omega_t \in \Omega,$, is $P(\cdot, \omega_t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We impose that both the absolute demand and choke price are finite: $D(0, \omega) < \infty$ and $P(0, \omega) < \infty$ for all $\omega \in \Omega$. It will be useful to denote $\bar{D} = \max_{\omega \in \Omega} D(0, \omega)$ and $\bar{P} = \max_{\omega \in \Omega} P(0, \omega)$.

Firm i's capacity $x_i(t) \in \mathbb{R}_+$ is the absolute limit on the number of units it can produce in period t. The marginal cost of production is zero up to the firm's capacity and infinite for any quantity beyond. The two firms have a common discount factor $\delta \in (0, 1)$.

The first two assumptions establish the basic properties of the industry demand function.

Assumption 1 *For each $\omega \in \Omega$ the quantity $D(0, \omega)$ is such that for all $q \in [0, D(0, \omega))$, $P(q, \omega) \in (0, \infty)$ and for all $q \geq D(0, \omega)$, $P(q, \omega) = 0$. On $(0, D(0, \omega))$, $P(q, \omega)$ is twice-continuously differentiable, strictly decreasing and concave in q.*

Assumption 2 *For all $\omega, \omega' \in \Omega$ such that $\omega > \omega'$, $D(p, \omega) \geq D(p, \omega')$ for all $p \in [0, P(0, \omega)]$.*

Since $D(p, \omega) \geq D(p, \omega')$ for all $p \in \mathbb{R}_+$, the state ω is considered to be a larger demand state than ω'.

The next two assumptions specify the properties of the firms' capacity cost functions.

Assumption 3 *Each firm's capacity cost function $c_i(\cdot)$ is twice-continuously differentiable, strictly increasing and convex.*

Assumption 4 *Positive profits are possible in any state $\omega \in \Omega$: For both firms $i \in \{1, 2\}$,*

$$c_i'(0) < \min_{\omega \in \Omega} P(0, \omega).$$

Let us establish some important notation. Label $\mathbb{P} = [0, \bar{P}] \times [0, \bar{P}]$ the set of viable pure prices for any period t. There is a pricing strategy contingent on the realization of ω. Hence, we define $p(\omega)$ the mixed pricing strategy based on the realization ω. While we denote by $\boldsymbol{p} = (p(\omega))_{\omega\in\Omega}$, the set of prices for all possible demand realizations. When referring to the pricing for a fixed state ω, we when it does not create confusion, we will often omit the argument and simply denote prices by $p = (p_1, p_2)$. With regards to capacities, we denote pure capacities by $x = (x_1, x_2) \in \mathbb{X} = [0, \bar{D}] \times [0, \bar{D}]$.

2.1 Rationing and Payoffs

Rationing of residual demand follows the proportional rule. An example where this rule appears reasonable is when demand is composed of a continuum of consumers that each demand one unit of the good.[8] The expected residual demand follows the proportional rule. The proportional rule is formally defined below.

$$D_i(p, x, \omega) = \begin{cases} \min\{D(p_i, \omega), x_i\} & \text{if } p_i < p_j \\ \min\left\{\max\left\{D(p_i, \omega)/2, D(p_i, \omega)\left(1 - x_j/D(p_j, \omega)\right)\right\}, x_i\right\} & \text{if } p_i = p_j \\ \max\left\{\min\left\{D(p_i, \omega)\left(1 - x_j/D(p_j, \omega)\right), x_i\right\}, 0\right\} & \text{if } p_i > p_j \end{cases}$$

Denote the state ω revenue of firm i by

$$V_i(p, x, \omega) = p_i D_i(p, x, \omega).$$

Denote a single period expected profit of firm i by

$$\mathcal{V}_i(\boldsymbol{p}, x) = E[V_i(p, x, \omega)] - c(x_i).$$

Further, denote by $V^J(p, x, \omega) = V_1(p, x, \omega) + V_2(p, x, \omega)$ and $\mathcal{V}^J(\boldsymbol{p}, x) = \mathcal{V}_1(\boldsymbol{p}, x) + \mathcal{V}_2(\boldsymbol{p}, x)$, the joint expected revenue for state ω and the joint expected profit, respectively.

2.2 Non-collusive equilibrium

The character of the Nash equilibrium in any individual pricing subgame will depend on the firms' previous capacity choices x. We appeal to Dasgupta and Maskin (1986b) for the existence of a mixed strategy Nash equilibrium in any pricing subgame. Our characterization of this equilibrium is based on results in Davidson and Deneckere (1986).

First let us describe the key features of Nash equilibrium of the simultaneous game as characterized in Davidson and Deneckere (1986). They have shown the existence of a unique mixed strategy equilibrium; we label the equilibrium strategies (ϕ_1^n, ϕ_2^n). It will be useful to

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[8] For an informative discussion of the prortional rationing scheme, see Davidson and Deneckere (1986).

denote the unique monopoly revenue for state ω by $\pi^M(\omega) = \max_{\rho\in[0,\bar{P}]} \rho D(\rho,\omega)$ and denote by $D^M(\omega)$, the monopoly quantity and $\rho^M(\omega)$ the monopoly price.

There are three distinct pricing regions determined by capacities:

$$\begin{aligned} C &= \left\{x \in \mathbb{R}^2_+ \;\mid x_1 + x_2 \leq D^M(\omega)\right\}, \\ M &= \left\{x \in \mathbb{R}^2_+ \;\mid x \notin C,\ \min\{x_1, x_2\} < D(0,\omega)\right\}, \\ B &= \left\{x \in \mathbb{R}^2_+ \;\mid \min\{x_1, x_2\} \geq D(0,\omega)\right\}. \end{aligned}$$

The Nash equilibrium expected revenue for firm a in each region is given by the function

$$V_i^n(x,\omega) = \begin{cases} P(x_1 + x_2, \omega)x & \forall x \in C \\ \pi^M(\omega)\int_{\underline{\rho}_i(x,\omega)}^{P(x_j,\omega)} \left(1 - x_j/D(z,\omega)\right) d\phi_j^n & \forall x \in M \text{ such that } x_i \geq x_j \\ \underline{\rho}_i(x,\omega)x_i & \forall x \in M \text{ such that } x_i < x_j \\ 0 & \forall x \in B \end{cases}. \tag{1}$$

The correspondence $\underline{\rho}_i(x,\omega)$ is the lower bound of the support of both firms' mixed strategies. This is implicitly defined by

$$\underline{\rho}_i(x,\omega) = \min\left\{\rho \in [0,\bar{P}] \,\middle|\, \rho \min\{D(\rho,\omega), x_j\} = \pi^m(\omega)\int_{\rho}^{P(x_i,\omega)} \left(1 - \frac{x_i}{D(z,\omega)}\right) d\phi_i^n \right\},$$

which is defined for all $x_i \geq 0$, and is twice continuously differentiable for all $x_i \in (0, D(\omega))$. When capacities and the state are fixed, it will be convenient to omit the arguments and denote $\widehat{\rho}_i = \max \underline{\rho}_i(x,\omega)$.

Denote by $\mathcal{V}_i^n(x) = E[V_i^n(x,\omega)] - c(x_i)$, firm i's non-collusive equilibrium expected profit.

2.3 Basic of collusive equilibria

We begin our analysis by establishing the collusive baseline of the joint profit maximizing solution. Denote $\rho^m(\omega)$ and x^m, the constrained joint profit maximizing price for any state ω and the joint profit maximizing capacities, respectively. The following proposition establishes the existence and uniqueness of the joint profit maximizing solution.

We impose that neither firm makes non-positive profit with joint profit maximizing prices and each firm has positive capacity.

Proposition 1 *The joint profit maximizing prices and capacities are* $\boldsymbol{p}^m = (\boldsymbol{\rho}^m, \boldsymbol{\rho}^m)$ *and* $x^m = (x_1^m, x_2^m)$ *such that* $x_1^m + x_2^m = \chi^m$ *described in 1 and 2.*

1. *For all* $\chi \in \mathbb{R}_+$ *and* $\omega \in \Omega$, *there exists a unique* $\rho^m(\omega)$ *such that*

$$\rho^m(\omega) = \begin{cases} \arg\max_{\rho \in [0,\bar{P}]} \rho D(\rho,\omega) & \text{if } \chi > D(\rho^m(\omega),\omega) \\ P(\chi,\omega) & \text{if } \chi \leq D(\rho^m(\omega),\omega) \end{cases} . \tag{2}$$

2. *Given pricing* $\boldsymbol{\rho}^m$, *the* x^m *such that*

$$x^m \in \arg\max_{x \in \mathbb{X}} \{E[\min\{\rho^m(\omega)D(\rho^m(\omega),\omega), x_1 + x_2\}] - c_1(x_1) - c_2(x_2)\} . \tag{3}$$

3 Collusive pricing

In this section, we focus on most-collusive equilibrium pricing with fixed capacities and discounts. We define *most-collusive pricing* as the joint profit maximizing pricing that is sustainable as a subgame perfect equilibrium. Throughout this section we assume, without loss of generality, that $x_1 \geq x_2$.We also take the expected punishment to firm i after a defection to be the arbitrary value $\mathcal{V}_i^P \geq 0$. When and if firms' can adjust capacity will effect this exact value. In terms of the pricing section to follow, all results are proved, without loss of generality, assuming arbitrary punishment values. In section 4, we will give context to this assumption.

As a prelude to the analysis of collusive pricing, we layout the incentive compatible constraints that determine if collusive prices are sustainable. Take any pricing strategy $\boldsymbol{p}$ and $p \in P \cup \phi^n$ and define the upper bound to the gain of defection from p and the future loss from punishment below. Notice that pricing can be any pure strategy prices as well as the non-collusive equilibrium prices (which can be in mixed strategies).

$$G_i(p,x,\omega) = \sup\nolimits_{\rho \in [0,\bar{P}]} V_i((\rho,p_j),x,\omega) - V_i(p,x,\omega), \ \forall p \in \mathbb{P} \cup \phi^n,$$

$$\mathcal{L}_i(\boldsymbol{p},x) = \frac{\delta}{1-\delta}\left(\mathcal{V}_i(\boldsymbol{p},x) - \mathcal{V}_i^P\right).$$

Define the set of prices that are incentive compatible in every states ω,

$$\Delta(x) = \left\{\boldsymbol{p} \in \prod\nolimits_{\omega \in \Omega} \mathbb{P} \cup \phi^n \left| G_i(p,x,\omega) \leq \mathcal{L}_i(\boldsymbol{p},x), \forall i \in \{1,2\} \ \forall \omega \in \Omega \right.\right\}.$$

It is worth noting that we can take the two future losses to be fixed constants $\mathcal{L}_1$ and $\mathcal{L}_2$. That is, any state price choice does not impact the future loss. This is a convenient consequence of the atomless distribution of uncertainty.

To help keep the concepts clear, we separately characterize both symmetric and asymmetric joint profit maximizing prices. First we address symmetric pricing.

3.1 Symmetric Pricing

It is possible, that for a given state ω, most-collusive pricing is symmetric. In this section, we restrict firms to symmetric pricing in any state ω and show that there always exists a unique joint profit maximizing symmetric price.

Denote by $G_1^s(\rho, x, \omega)$, the maximal gain from a defection of firm i from the symmetric price $\rho \in [0, \bar{P}]$ in state ω with capacities x. Let us define the upper bound to the gain of defecting from the symmetric price ρ by

$$G_i^s(\rho, x, \omega) = \rho \min\{D(\rho, \omega), x_1\} - \rho \min\left\{x_i, \max\left\{\frac{D(\rho,\omega)}{2}, D(\rho,\omega)\left(1 - \frac{x_j}{D(\rho,\omega)}\right)\right\}\right\}.$$

It is immediate that when $x_1 + x_2 \leq \chi^m$, $G_i^s(P(x_1 + x_2, \omega), x, \omega) = 0$; joint profit maximizing pricing is sustainable. Therefore, we only need to consider most-collusive symmetric pricing for capacities x such that case that $x_1 + x_2 > \chi^m$.

Denote by $\rho^{si}(\omega)$, the maximum price in $[0, \rho^m(\omega)]$ such that $G_i^s(\rho^{si}(\omega), x, \omega) = \mathcal{L}_i$.

Proposition 2 *If pricing is symmetric, then there exists unique symmetric pricing* $p^s(\omega) = (\rho^s(\omega), \rho^s(\omega))$ *such that*

$$\rho^s(\omega) = \begin{cases} \rho^m(\omega) \\ \min\left\{\rho^{s1}(\omega), \rho^{s2}(\omega)\right\} \end{cases}.$$

Before we begin the proof of the proposition we establish a preliminary result. The lemma states that a firm's maximal gain from defection is strictly increasing in the symmetric collusive price up to the joint profit maximizing price $\rho^m(\omega)$. This fact is the basis for our uniqueness argument in Proposition 2.

Lemma 1 *Take* ρ, ρ' *in* $[0, \rho^m(\omega)]$ *such that* $x_1 > D(\rho', \omega)/2$, *then*

$$G_1^s(\rho', x, \omega) > G_1^s(\rho, x, \omega) \quad \Longleftrightarrow \quad \rho' > \rho.$$

The proof of Proposition 2 now follows readily.

Proof of Proposition 2. First we address the existence of a maximum. Based on the atom-less distribution of uncertainty, no choice of an individual state price will negatively effect another state price. Therefore, the problem can be broken into a continuum of single state maximization problems with future prices fixed. Note that for all x, the joint expected revenue is bounded and strictly increasing in ρ on $[0, \rho^m(\omega)]$. Thus, combined with Lemma 1, we know the joint profit maximizing symmetric prices for any state ω must solve

$$\rho^s(\omega) \in \max\{\rho \in [0, \rho^m(\omega)] \mid G_i^s(\rho, x, \omega) \leq \mathcal{L}_i, \forall i \in \{1, 2\}\}. \tag{4}$$

The constraint set is composed of linear weak inequalities of functions continuos in ρ on $[0, \rho^m(\omega)]$, a sufficient condition for the constraint set to closed. By construction $\{\rho \in [0, \rho^m(\omega)] \mid G_i^s(\rho, x, \omega) \leq \mathcal{L}_i, \forall i \in \{1, 2\}\}$ is a subset of the compact space $[0, \rho^m(\omega)]$, and a closed subset of a compact space is compact. If the constraint set is non-empty, then it must have a maximal element.

Now we move to the question of uniqueness. There are four cases of different prices ρ^s:

(i) $G_1^s(\rho^s, x, \omega) = \mathcal{L}_1$ and $G_2^s(\rho^s, x, \omega) < \mathcal{L}_2$,

(ii) $G_1^s(\rho^s, x, \omega) < \mathcal{L}_1$ and $G_2^s(\rho^s, x, \omega) = \mathcal{L}_2$,

(iii) $G_1^s(\rho^s, x, \omega) = \mathcal{L}_1$ and $G_2^s(\rho^s, x, \omega) = \mathcal{L}_2$, or

(iv) $G_1^s(\rho^s, x, \omega) < \mathcal{L}_1$ and $G_2^s(\rho^s, x, \omega) = \mathcal{L}_2$.

In all cases (i)-(iii), based on Lemma 1 and the fact that $\mathcal{L}_1$ is independent of ρ (based on the atom-less distribution of uncertainty), the maximal price must be unique. While for case (iv), trivially, $\rho^s = \rho^m$. ■

3.2 Asymmetric Pricing

The purpose of this section is to characterize most-collusive asymmetric pricing. As in Section 3.1, its only necessary to consider asymmetric pricing for capacities such that $x_1 + x_2 > D^m(\omega)$. Otherwise, constrained monopoly pricing is always sustainable and leads to the greatest possible joint revenue for that state. First we show that there are only two possible most-collusive asymmetric prices, one possibility with firm 1 pricing greater and the other with firm 2 pricing greater.

The upper bound on the gain from defecting for the higher price firm (firm i) is

$$G_i^a(p, x, \omega) = p_j \min\{D(p_j, \omega), x_i\} - p_i \min\left\{0, D(p_i, \omega)\left(1 - \frac{x_j}{D(p_j, \omega)}\right)\right\}, \tag{5}$$

The upper bound on the gain from defecting for the lower price firm (firm j) is

$$g_j^a(p, x, \omega) = p_i \min\{D(p_i, \omega), x_j\} - p_j x_j. \tag{6}$$

In what follows, we prove that the unique solution to the two maximization problems for $i = 1, 2$ defined in (7) describe the two possible types of most-collusive asymmetric pricing.

$$p^{ai} \in \arg\max_{p \in \mathbb{P}} \left\{ V^J(p, x, \omega) \,\middle|\, G_i^a(p, x, \omega) \leq \mathcal{L}_i,\ g_j^a(p, x, \omega) \leq\ \mathcal{L}_j, \text{ and } \rho^m \geq p_i \geq p_j \right\}. \tag{7}$$

Pricing $p_i = p_j$ is included in the maximization problem despite that fact that such pricing is symmetric. This is done to close the maximization problem and does not adversely effect our characterization of most-collusive pricing in Section 3.3.

The solution to the problem in (7) can be examined in two sequential steps. The first step is choosing p_j to maximize joint revenue based on a fixed p_i and subject to the constraint $G_i^a(p,x,\omega) \leq \mathcal{L}_i$. Let use denote this solution by $\widetilde{p}_j(p_i)$, where for all $p_i \in [\widehat{\rho}_2, \rho^m]$,

$$\widetilde{p}_j(p_i) = \arg \max_{p_j \in [\widehat{\rho}_2, p_i]} \{V^J(p,x,\omega) \mid G_i^a(p,x,\omega) \leq \mathcal{L}_i\}. \tag{8}$$

Lemma 2 *If $\mathcal{L}_i > 0$, then there exists a unique solution $\widetilde{p}_j(p_i)$, which is weakly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$. Further, $\widetilde{p}_j(p_i) = p_i$ or $\widetilde{p}_j(p_i)$ is such that $G_i^a(\widetilde{p}_j(p_i), p_i, x, \omega) = \mathcal{L}_i$.*

The lemma establishes that problem (8) always has a boundary solution; $\widetilde{p}_j(p_i)$ is the maximum price in the constraint set, which is the minimum of p_i and the price that equates firm j's gain from defection to their expected loss from future punishment.

The second component of most-collusive asymmetric pricing is firm i's choice of p_i given that firm j's price is determined by $\widetilde{p}_j(p_i)$. We denote by $\widetilde{p}_i^{ai}$ the solution to the following program:

$$\widetilde{p}_i^{ai} = \arg \max_{p_i \in [\widehat{\rho}_2, \rho^m]} \{V^J(\widetilde{p}_j(p_i), p_i, x, \omega) \mid g_j^a(\widetilde{p}_j(p_i), p_i, x, \omega) \leq \mathcal{L}_j\}. \tag{9}$$

Lemma 3 *If $\mathcal{L}_1, \mathcal{L}_2 > 0$, then there exists a unique solution $\widetilde{p}_i^{ai}$ such that, $\widetilde{p}_i^{ai} = \rho^m$ or $\widetilde{p}_i^{ai}$ is such that $g_j^a(\widetilde{p}_j(\widetilde{p}_i^{ai}), \widetilde{p}_i^{ai}, x, \omega) = \mathcal{L}_j$.*

Again (analogous to Lemma 2), Lemma 3 shows that the solution is at the largest price in the constraint set. This is the minimum of the joint profit maximizing price and the price that equates firm i's gain from defection to its expected loss from punishment.

Lemma 2 and 3 form the basis for following proposition regarding most-collusive asymmetric pricing.

Proposition 3 *The most-collusive asymmetric pricing is*

$$p^a = \arg\max\nolimits_{p \in \{p^{a1}, p^{a2}\}} V^J(p,x,\omega),$$

where $p^{ai} = (\widetilde{p}_i^{ai}, \widetilde{p}_j(\widetilde{p}_i^{ai}))$ for $i \in \{1,2\}$.

The proof establishes that the unique solution to (7) is $p^{ai} = (\widetilde{p}_i^{ai}, \widetilde{p}_j(\widetilde{p}_i^{ai}))$ for $i \in \{1,2\}$. It immediately follows that most-collusive asymmetric pricing is the maximizer of joint expected revenue between p^{a1} and p^{a2} .

Proof of Proposition 3. We show that p^{ai} is the unique solution to (7). Suppose the contrary that p^{ai} is not the unique solution to (7). In other words there exists a $p' \neq p^{ai}$ that is incentive compatible and such that $V^J(p', x, \omega) \geq V^J(p^{ai}, x, \omega)$. We proceed by showing that such a p must violate one of the incentive constraints.

Since the function $V^J(p, x, \omega)$ is increasing in p for prices less than p^m, we can conclude that at least one firms price must be greater than there price in p^{ai} for p' to be such that $V^J(p', x, \omega) \geq V^J(p^{ai}, x, \omega)$ to be true. Thus, we have two cases: $p'_i \in (p^{ai}_i, \rho^m]$, or $p'_j \in (p^{ai}_j, p_i]$.

First, we take $p'_i \in (p^{ai}_i, \rho^m]$. Notice that $p'_j > \widetilde{p}_j(p'_i)$, because $(p'_i, \widetilde{p}_j(p'_i))$ cannot lead to revenue as high at p^{ai}, or p'_i is a solution to program (9), which it is not. The proof Lemma 2 property (ii) established that V^J is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$. Thus, if $p'_j > \widetilde{p}_j(p'_i)$, then p'_j leads to more expected revenue than $\widetilde{p}_j(p'_i)$, which cannot be true. Thus, p' must violate the incentive compatibility condition in (5).

Second, we take $p'_j \in (p^{ai}_j, p_i]$. Since $p^{ai} = (\widetilde{p}_j(\widetilde{p}^{ai}_i), \widetilde{p}^{ai}_i)$ we know that

$$\begin{aligned}
\mathcal{L}_i &= G^a_i(p^{ai}, x, \omega) \\
&= p^{ai}_j \min\{D(p^{ai}_j, \omega), x_i\} - p^{ai}_i \min\left\{0, D(p^{ai}_i, \omega)\left(1 - \frac{x_j}{D(p^{ai}_j, \omega)}\right)\right\} \\
&< p'_j \min\{D(p'_j, \omega), x_i\} - p^{ai}_i \min\left\{0, D(p^{ai}_i, \omega)\left(1 - \frac{x_j}{D(p'_j, \omega)}\right)\right\}.
\end{aligned}$$

To make $G^a_i(p', x, \omega) \leq \mathcal{L}_i$, the price p'_i must be such that $p'_i > p^{ai}_i$ because $G^a_i(p, x, \omega)$ is strictly decreasing in p_i. Thus, it must be that $p' \gg p^{ai}$. We have established in the proof of Lemma 3 that if $p' \gg p^{ai}$, then $V^J(p', x, \omega) > V^J(p^{ai}, x, \omega)$.

Now we examine the incentive constraint (6). We know that $V^J(p^{ai}, x, \omega) > V^J(p'_i, \widetilde{p}_j(p'_i), x, \omega)$, otherwise p^{ai}_i could not be the unique solution to (9). Combining the inequalities in the two previous sentences we have $V^J(p', x, \omega) > V^J(p'_i, \widetilde{p}_j(p'_i), x, \omega)$. But $\widetilde{p}_j(p'_i)$ is the unique solution to (8). Thus, p'_j must violate (6), which contradicts p' as a solution to (7). ■

3.3 Most-collusive pricing

We now move to the characterization of most-collusive equilibrium pricing. The pricing results apply to many semi-collusive equilibria (with non-optimal capacities) as well as the full most-collusive equilibria. It is the size of the demand state relative to the capacities and the discount that determines most-collusive pricing.

First, we establish that for large enough discount factors monopoly pricing is sustainable in all demand states, and that for small enough discounts only non-collusive pricing is sustainable in all demand states.

Lemma 4 *For any fixed x, there exists $\overline{\delta}$ and $\underline{\delta}$ such that the most-collusive prices follow: $\boldsymbol{p}^* = \boldsymbol{p}^m$, $\forall \delta \in [\overline{\delta}, 1)$; and $\boldsymbol{p}^* = \boldsymbol{\phi}^n$, $\forall \delta \in (0, \underline{\delta}]$.*

Based on Sections 3.1 and 3.2, for we know that between $\overline{\delta}$ and $\underline{\delta}$ pricing in state ω is either the most-collusive symmetric pricing or most collusive asymmetric pricing. It is important now that we make the distinction that pricing is asymmetric, $p = p^a$ only if, $p = p^a \neq p^s$, prices are actually asymmetric. Thus, in the statement of the following theorem this is the meaning of p^a or consequently p^{a1}. For different special cases, the character of pricing between $\overline{\delta}$ and $\underline{\delta}$ is well behaved. Three different cases are described in the following theorem.

Theorem 1 *Most collusive pricing follows (1)-(4) below.*

1. *For all x such that $x_2 \in (D^m(\omega)/2, D(\widehat{\rho}_2, \omega)/2)$,*
    (a) *there exists $\delta^a \in (\underline{\delta}, \overline{\delta})$ such that*

    $$p^*(\omega) \in \begin{cases} p^s(\omega) \cup p^a(\omega) & \forall \delta \in [\delta^a, \overline{\delta}) \\ p^a(\omega) & \forall \delta \in (\underline{\delta}, \delta^a) \end{cases} ;$$

    (b) *and $x_1 \geq D(\widehat{\rho}_2, \omega)$, there exists $\delta^{a1} \in (\underline{\delta}, \overline{\delta})$ such that*

    $$p^*(\omega) \in \begin{cases} p^s(\omega) \cup p^{a1}(\omega) & \forall \delta \in [\delta^{a1}, \overline{\delta}) \\ p^{a1}(\omega) & \forall \delta \in (\underline{\delta}, \delta^{a1}) \end{cases} .$$

2. *For all x such that $x_2 \leq D^m(\omega)/2$*
    (a) *$p^*(\omega) = p^a(\omega)$ $\forall \delta \in (\underline{\delta}, \overline{\delta})$;*
    (b) *and $x_1 \geq D(\widehat{\rho}_2, \omega)$, $p^*(\omega) = p^{a1}(\omega)$ $\forall \delta \in (\underline{\delta}, \overline{\delta})$.*
3. *For all x such that $x_2 \geq D(\widehat{\rho}_2, \omega)$, $p^*(\omega) = p^s(\omega)$ $\forall \delta \in (\underline{\delta}, \overline{\delta})$.*
4. *For all x such that $x_1 + x_2 \leq D^m(\omega)$, $p^*(\omega) = p^m(\omega)$ $\forall \delta \in (\underline{\delta}, \overline{\delta})$.*

Proof of Theorem 1 The proof of the theorem is based on the series of results, Lemmas 5-7.

Lemma 5 *For all $x \in \mathbb{X}$ and $\delta \in (\underline{\delta}, \overline{\delta})$, if p^s is such that $x_2 \leq D(\rho^s, \omega)/2$, then $p^* = p^a$. Further, take δ^a such that $x_2 = D(\rho^s, \omega)/2$, then for all $\delta \in (\underline{\delta}, \delta^a)$, $p^* = p^a$.*

Proof of Lemma 5. The first part of the proof is composed by way of comparison of the gains from defection at the price ρ^s. The symmetric gain is:

$$G_1^s(\rho^s, x, \omega) = \rho^s \min\{D(\rho^s, \omega), x_1\} - \rho^s D(\rho^s, \omega) \left(1 - \frac{x_2}{D(\rho^s, \omega)}\right).$$

While at a prices ρ^s and $\rho^m > \rho^s$:

$$G_1^a(\rho^m, \rho^s, x, \omega) = \rho^s \min\{D(\rho^s, \omega), x_1\} - \rho^m D(\rho^m, \omega)\left(1 - \frac{x_2}{D(\rho^s, \omega)}\right)$$

Since $\rho^m D(\rho^m, \omega) > \rho^s D(\rho^s, \omega)$, $G_1^a(\rho^m, \rho^s, x, \omega) < G_1^s(\rho^s, x, \omega)$.

The second firm's gain is:

$$g_2^a(\rho^m, \rho^s, x, \omega) = (\rho^s - \rho^m)\, x_2 < 0.$$

Hence $p_1 = \rho^m > \rho^s$ and $p_2 = \rho^s$ has a lower gain from defection for both firms. Since uncertainty follows an atom-less distribution, the expected loss from defection is fixed. Therefore, the asymmetric pricing is sustainable and leads to higher revenue joint revenue . Thus, symmetric pricing must not be optimal.

The second part of the proof follows from that fact that ρ^s is decreasing in δ by way of the losses decreasing in δ. Hence, if $x_2 = D(\rho^s, \omega)/2$ at δ^a, then for all $\delta \in (\underline{\delta}, \delta^a)$, $x_2 < D(\rho^s, \omega)/2$. ■

Lemma 6 *For all $x \in \mathbb{X}$ such that $x_1 \geq D(\widehat{\rho}_2, \omega)$, $p_1^* \geq p_2^*$, $\forall \delta \in (\underline{\delta}, \overline{\delta})$.*

Proof of Lemma 6. We suppose to the contrary that $p_1^*(\omega) < p_2^*(\omega)$ and show a contradiction. We do this by comparing the gain from a defection for firm 2 at $\widehat{\rho}_2 \leq p_1 = \rho < p_2$ verse both firms pricing at ρ. The gains are (note the condition $x_1 \geq D(\widehat{\rho}_2, \omega)$ is used to determine the gains from an asymmetric defection):

$$\begin{aligned} G_2((p_2, \rho), x, \omega) &= \rho \min\{D(\rho, \omega), x_2\}, \\ G_2^s(\rho, x, \omega) &= \rho \min\{D(\rho, \omega), x_2\} - \rho \min\{D(\rho, \omega)/2, x_2\}. \end{aligned}$$

Clearly, $G_2^s(\rho, x, \omega) < G_2((p_2, \rho), x, \omega)$, and hence $\rho^s > p_1^{a2}$. Since $x_1 \geq D(\widehat{\rho}_2, \omega)$, $V^J(p_1^{a2}, p_2) = p^{a2} D(p_1^{a2}, \omega)$. While at the symmetric prices p^s, $V^J(p^s) = \rho^s D(\rho^s, \omega)$. Since $\rho^m > \rho^s > p_1^{a2}$ and $\rho D(\rho, \omega)$ is strictly increasing on $[0, \rho^m]$, $V^J(p^s) > V^J(p_1^{a2}, p_2)$, a contradiction. ■

Lemma 7 *For all $x \in \mathbb{X}$ such that $x_1 \geq D(\widehat{\rho}_2, \omega)$ and $x_2 \leq D^m(\omega)/2$, $p^* = p^{a1} \neq p^s$, $\forall \delta \in (\underline{\delta}, \overline{\delta})$.*

Proof of Lemma 7. From Lemma 6 we know that for all $\delta \in (\underline{\delta}, \overline{\delta})$ most-collusive pricing is either p^s or p^{a1}. Lets suppose that it is p^s and show a contradiction. The gains from a defect from the symmetric price $\rho > \widehat{\rho}_2$ is (if this condition were not true, then the joint revenue would be less than that from the non-collusive prices ϕ^n):

$$\begin{aligned} G_1^s(\rho, x, \omega) &= \rho \min\{D(\rho, \omega), x_1\} - \rho D(\rho, \omega)\left(1 - \frac{x_2}{D(\rho, \omega)}\right). \\ G_2^s(\rho, x, \omega) &= 0 \end{aligned}$$

Now we look at the asymmetric pricing gains from $p' = (p_1, \rho^s)$ such that $p_1 > \rho^s$.

$$\begin{aligned} G_1^a((p_1,\rho^s),x,\omega) &= \rho^s \min\{D(\rho^s,\omega),x_1\} - p_1 D(p_1,\omega)\left(1 - \frac{x_2}{D(\rho^s,\omega)}\right). \\ g_2^a((p_1,\rho^s),x,\omega) &= (p_1 - \rho^s)\,x_1 \end{aligned}$$

For any $p_1 \in (\rho^s, \rho^m]$, $G_1^a((p_1,\rho^s),x,\omega) < G_1^s(\rho,x,\omega) = \mathcal{L}_1$. It must be that $\mathcal{L}_2 > 0$ (otherwise $\delta \le \underline{\delta}$), which implies that all $p_1 \le \mathcal{L}_2/x_1 + \rho^s$ are incentive compatible. Hence, there is a $p_1 \in (\rho^s, \rho^m]$ such that p' is incentive compatible. For symmetric pricing to be optimal $V^J(p^s) \ge V^J(p')$. Thus,

$$\begin{aligned} \rho^s x_2 + \rho^s D(\rho^s,\omega)\left(1 - \frac{x_2}{D(\rho^s,\omega)}\right) &\ge \rho^s x_2 + p_1 D(p_1,\omega)\left(1 - \frac{x_2}{D(\rho^s,\omega)}\right), \\ \rho^s D(\rho^s,\omega) &\ge p_1 D(p_1,\omega). \end{aligned}$$

The final inequality is a contradiction based on the fact that $\rho D(\rho,\omega)$ is strictly increasing on $[0,\rho^m]$. ■

Statement 1(a) follow

The statements of the theorem now follow readily from Lemmas 5-7. Statement 1(a) is a direct consequence of Lemma 5, while 1(b) follows from Lemmas 5&6. Statement 2(a) is a special cases of 1(a)that also follows from Lemma 5, while 2(b) follows directly from Lemmas 7.

Statement 3 is follows from Lemma 6. Since $x_2 \ge D(\widehat{\rho}_2,\omega)$, we know from Lemma 6 that $p_2^* \ge p_1^*$. We have imposed that $x_1 \ge x_2$, which implies that $x_1 \ge D(\widehat{\rho}_2,\omega)$ and from Lemma 6, $p_1^* \ge p_2^*$. Therefore, when $x_2 \ge D(\widehat{\rho}_2,\omega)$, $p_1^* = p_2^*$.

The final statement of the theorem is an immediate consequence of the joint profit maximizing prices yielding no gain from defection. ■

3.4 Numerical Examples

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4 Collusive Capacities and Prices

In terms of capacities, we focus on the case that a firm's capacity depreciates fully after one period and firms choose new capacities simultaneously at the beginning of each period. This is the model specification used by Staiger and Wolak (1992).

Punishment for a defection in price: We assume that the punishment to a price defection, which we denote $\mathcal{V}_i^P$, is non-negative number between min-max and noncooperative reversion. Since the game is symmetric and pricing always concludes the stage-game, the punishment begins at a new stage game independent of the previous stage.

Punishment for capacity defections: We assume that a capacity defection is responded to with an immediate reversion to noncooperative pricing, which results in revenue between min-max and non-collusive pricing. Then starting at the new stage game the punishment is $\mathcal{V}_i^P$.

The arbitrary mixed pricing strategy $x \in \mathbb{X}$ is in the set of incentive compatible collusive prices if it is such that, where,

$$\mathcal{G}_i(\boldsymbol{p}, x) = \sup_{\chi \in [0, \bar{D}]} \mathcal{V}_i^P(\chi, x_j) - \mathcal{V}_i(\boldsymbol{p}, x),$$

Formally, define the set of capacities that satisfy the constraint set

$$\Phi(\boldsymbol{p}) = \left\{ x \in \mathbb{X} \,|\mathcal{G}_i(\boldsymbol{p}, x) \leq \mathcal{L}_i(\boldsymbol{p}, x), \, \forall i \in \{1, 2\} \right\}. \tag{10}$$

Collusive capacities can be any capacities $x \in \Phi(\boldsymbol{p}^*)$. For sufficiently large discount factors, the set of collusive pure strategy capacities is non-empty. For fairly large discount factors, the set can include many symmetric and asymmetric capacity combinations. The most-collusive capacities are the joint profit maximizing capacities in the set $\Phi(\boldsymbol{p}^*)$. If the discount factor is sufficiently such that $\boldsymbol{p}^* = \boldsymbol{p}^m$ for all $x \in \mathbb{X}$, then the set of most-collusive capacities is the set of joint profit maximizing capacities.

5 Conclusion

6 Appendix

Proof of Proposition 1. The proof is done in two parts beginning with statement 1 in the proposition .

Part 1 (Pricing):

Existence: Both revenue functions are continuous in p on $\mathbb{P}$ therefore the joint revenue is continuous in p on $\mathbb{P}$. The set $\mathbb{P}$ is compact and a continuous function on a compact set achieves a maximum.

Uniqueness: First, we show that: among symmetric prices there is a unique joint profit maximizing solution. We must argue that, for symmetric prices, the joint profit function is strictly concave. Based on the assumptions 1-4, for all χ, we show there is a unique solution to the problem constrained maximization problem

$$\rho^m(\omega) = \arg\max_{\rho\in[0,\bar{P}]} \{\rho D(\rho,\omega) | D(\rho,\omega) \geq \chi\} \text{ for all } \omega \in \Omega \tag{11}$$

for any subgame. If ω is such that $D(\rho^M(\omega),\omega) < \chi$, then the solution to the maximization (11) problem is the solution to the unconstrained problem $\rho^M(\omega)$, and the revenue is uniquely equal to π^M, since no other revenue can be greater. If the constraint binds, then the highest price that uses all the capacity gives the most joint revenue. Since $\rho D(\rho)$ is strictly concave, $\rho D(\rho)$ is strictly decreasing on $[\rho^M(\omega), \bar{P}]$. Therefore, $\widetilde{\rho}$ such that $D(\widetilde{\rho},\omega) = \chi$ has higher revenue than any greater price. Any lower price sells to the same demand χ at a lower price. Hence, the symmetric joint profit maximizing price is uniquely determined by this equality by inverting the demand function $\rho^m(\omega) = P(\chi,\omega)$. Thus, the monopoly revenue of any period t is given by

$$\overline{V}^J(\chi,\omega) = \begin{cases} \pi^M & \text{if } D(\rho^M,\omega) < \chi \\ P(\chi,\omega)\chi & \text{if } D(\rho^M,\omega) \geq \chi. \end{cases} \tag{12}$$

Next we show that: no asymmetric prices give higher joint revenue than ρ^m.We will show, case by case, that any asymmetric pricing does not maximize joint profit. Without loss of generality lets impose that $p_i < p_j$.

Case 1: ($x_1 > D^M(\omega)$ and $p_j > \rho^M$). In this case $\rho^m = \rho^M$. $V^J(p,x,\omega) = p_i D(p_i,\omega)$. Instead if firm i prices at ρ^M the joint profit is π^M, which is greater than $p_i D(p_i,\omega)$ by definition of π^M. Since, neither firm can have zero profit, firm j can price only at ρ^M and still the joint profit is π^M.

Case 2: ($x_1 > D^M(\omega)$ and $p_j < \rho^M$). In this case $\rho^m = \rho^M$. $V^J(p,x,\omega) = p_i D(p_i,\omega)$. Instead if both firms price at ρ^M the joint profit is π^M, which is greater than $p_i D(p_i,\omega)$ by definition of π^M. Since, neither firm can have zero profit, firm j can price only at ρ^M, not higher.

Case 3:($p_j \neq \rho^m$, $x_1 \leq D^M(\omega)$, and $x_1 + x_2 \geq D^m(\omega)$).

Case 3(i): The joint profit of the firms is $V^J(p,x,\omega) = p_i x_i + p_j D(p_j)(1 - x_i/D(p_i))$ If its firm j prices at ρ^M the joint profit is

$$\begin{aligned} V^J(p_i,\rho^M,x,\omega) &= p_i x_i + \pi^M(1 - x_i/D(p_i)) \\ &= \pi^M + \underbrace{\left(\frac{p_i D(p_i) - \pi^M}{D(p_i)}\right)}_{(A)} x_i. \end{aligned}$$

For all $p_i \neq \rho^M$, $A < 0$ and the joint profit is less than π^M. At $p_i = \rho^M$, $A = 0$ and the joint profit is π^M.

Case 3(ii): The joint profit of the firms is $V^J(p, x, \omega) = p_i x_i + p_j x_j$. In this case $p_j < \rho^M$ (because demand is decreasing in price). If firm j prices at ρ such that $D(\rho, \omega)(1 - x_i/D(p_i, \omega) = x_j$, the joint profit increases. Further, it must be that $\rho < \rho^M$. Hence (p_i, ρ), are prices in Case 3(i).

Case 4: $(p_j \neq \rho^m$, $x_1 \leq D^M(\omega)$, and $x_1 + x_2 < D^m(\omega))$.

Case 4(i): If $p_j > \rho^m$, then residual profit decreases for all prices in the interval $[p_i, p_j]$. Thus, firm j pricing at p_i increases joint profit. Since residual profit is $p_j D(p_j)(1 - x_i/D(p_i))$.

Case 4(i).a: If $p_i > \rho^m$, then firm j pricing at p_i increases joint profit. Since residual profit is $p_j D(p_j)(1 - x_i/D(p_i))$.

$$V^J(p_i, \rho^m, x, \omega) = p_i x_i + \rho^m x_j < \rho^m(x_1 + x_2) = V^J(p^m, x, \omega).$$

Case 4(i).b: If $p_i < \rho^m$, then the joint profit can be increased by setting $p_j = \rho^m$, since the residual demand is decreasing on $[\rho^m, p_j]$. Further, firm i can increase joint profit by switching to ρ^m:

$$V^J(p_i, \rho^m, x, \omega) = p_i x_i + \rho^m x_j < \rho^m(x_1 + x_2) = V^J(p^m, x, \omega).$$

Case 4(ii) If $p_j < \rho^m$, then the joint profit is $V^J(p, x, \omega) = p_i x_i + p_j x_j$. If firm i prices at $p_i = p_j$, then the joint profit increase to $p_j(x_1 + x_2)$.

Case 5(i): $(p_i < p_j = \rho^m$, $x_1 \leq D^M(\omega)$ and $x_1 + x_2 \geq D^m(\omega))$. If its firm j prices at ρ^M the joint profit is

$$\begin{aligned} V^J(p_i, \rho^M, x, \omega) &= p_i x_i + \pi^M(1 - x_i/D(p_i)) \\ &= \pi^M + \underbrace{\left(\frac{p_i D(p_i) - \pi^M}{D(p_i)}\right)}_{(A)} x_i. \end{aligned}$$

For all $p_i \neq \rho^M$, $A < 0$ and the joint profit is less than π^M. At $p_i = \rho^M$, $A = 0$ and the joint profit is π^M.

Case 5(ii): $(p_i < p_j = \rho^m$, $x_1 \leq D^M(\omega)$ and $x_1 + x_2 < D^m(\omega))$. Firm i can increase joint profit by switching to ρ^m:

$$V^J(p_i, \rho^m, x, \omega) = p_i x_i + \rho^m x_j < \rho^m(x_1 + x_2) = V^J(p^m, x, \omega).$$

Part 2 (capacities):

Existence: In the first part of the proof we show there exists a joint maximal capacity solution. It is straight forward from (12) to see that $\overline{V}^J(\chi, \omega)$ is bounded for all $\chi \geq 0$.

Denote the joint profit with pricing $\boldsymbol{\rho}^m$ as a function of capacities $\overline{\mathcal{V}}^J(x) = E[\overline{V}^J(x_1+x_2,\omega)] - c_1(x_1) - c_2(x_2)$. Because each $\overline{V}^J(x_1+x_2,\omega)$ is bounded, the expectation $E[\overline{V}^J(x_1+x_2,\omega)]$ is also bounded. Since both costs are bounded on $\mathcal{X}$, the entire expression $\overline{\mathcal{V}}^J(x)$ is bounded on $\mathcal{X}$. Since $\lim_{\chi\downarrow D^M} P(\chi,\omega)\chi = \pi^M$ and $P(\chi,\omega)\chi$ is continuous in χ, each state joint revenue function is continuous. A expectation $(E[\overline{V}^J(x_1+x_2,\omega)])$ over bounded continuous functions is itself continuous. Combined with the continuity of the cost functions, the function $\overline{\mathcal{V}}^J(x)$ is continuous. By assumption $\mathcal{X}$ is compact. A maximum exists for a continuous bounded function on a compact set. ■

Proof of Lemma 1. The proof that $G_1^s(\rho',x,\omega) > G_1^s(\rho,x,\omega)$ if $\rho' > \rho$ is done by, straight forward, case by case comparison of these gains. We break the proof into four cases.

Case 1: If $x_1 \le D(\rho,\omega)/2$, then the respective gains are:

$$\begin{aligned} G_1^s(\rho,x,\omega) &= 0, \text{ and} \\ G_1^s(\rho',x,\omega) &= \rho' \left(\min\{x_1, D(\rho',\omega)\} - D(\rho',\omega)/2\right) > 0. \end{aligned}$$

Case 2: If $x_1 + x_2 \ge D(\rho,\omega)$ and $x_2 < D(\rho,\omega)/2$, the then the respective gains are:

$$\begin{aligned} G_1^s(\rho,x,\omega) &= \rho x_2, \text{ and} \\ G_1^s(\rho',x,\omega) &= \rho' \min\{x_2, D(\rho',\omega)/2\}. \end{aligned}$$

First notice that if $x_2 \le D(\rho',\omega)/2$, then $G_1^s(\rho',x,\omega) = \rho' x_2 > \rho x_2$. While, if $x_2 > D(\rho',\omega)/2$, then $G_1^s(\rho',x,\omega) = \rho' D(\rho',\omega)/2$. Based on the strict concavity of $\rho D(\rho,\omega)$ and the fact that $\rho^m(\omega)$ is the unique maximizer of this function, $\rho D(\rho,\omega)$ is strictly increasing on $[0,\rho^m(\omega)]$. Thus, $\rho' D(\rho',\omega)/2 > \rho D(\rho,\omega)/2$. Since $D(\rho,\omega) > x_2$, $\rho D(\rho,\omega)/2 > \rho x_2$. Putting these inequalities together we have $G_1^s(\rho',x,\omega) > G_1^s(\rho',x,\omega)$.

Case 3: If $x_1 + x_2 \ge D(\rho,\omega)$ and $x_2 \ge D(\rho,\omega)/2$, then the respective gains are:

$$\begin{aligned} G_1^s(\rho,x,\omega) &= \rho D(\rho,\omega)/2, \text{ and} \\ G_1^s(\rho',x,\omega) &= \rho' D(\rho',\omega)/2. \end{aligned}$$

Based on the fact that $\rho D(\rho,\omega)$ is strictly increasing on $[0,\rho^m(\omega)]$, $G_1^s(\rho',x,\omega) = \rho' D(\rho',\omega)/2 > \rho D(\rho,\omega)/2 = G_1^s(\rho,x,\omega)$.

Case 4: If $D(\rho,\omega) > x_1 > x_2 \ge D(\rho,\omega)/2$, then the respective gains are:

$$\begin{aligned} G_1^s(\rho,x,\omega) &= \rho \left(x_1 - D(\rho,\omega)/2\right), \text{ and} \\ G_1^s(\rho',x,\omega) &= \rho' \left(\min\{x_1, D(\rho',\omega)\} - D(\rho',\omega)/2\right). \end{aligned}$$

If $x_1 \le D(\rho',\omega)$, then $G_1^s(\rho',x,\omega) = \rho' \left(x_1 - D(\rho',\omega)/2\right)$. This is clearly greater than $G_1^s(\rho,x,\omega)$, since both $\rho' > \rho$ and $x_1 - D(\rho',\omega)/2 > x_1 - D(\rho,\omega)/2$. If $x_1 > D(\rho',\omega)$, then $G_1^s(\rho', x, \omega) = \rho' D(\rho', \omega)/2$. Again, the fact that $\rho D(\rho, \omega)$ is strictly increasing on $[0, \rho^m(\omega)]$ implies that $\rho^{'} D(\rho', \omega)/2 > \rho D(\rho, \omega)/2$. Notice that $\rho D(\rho, \omega)/2 > G_1^s(\rho, x, \omega)$ because $x_1 < D(\rho, \omega)$. Hence, $G_1^s(\rho', x, \omega) > G_1^s(\rho, x, \omega)$. ■

Proof of Lemma 2. The main content of the proof of Lemma 2 is establishing four key properties (recall that it has been imposed that $p_j \le p_i \le \rho^m$): (i) V^J is continuous in p_j on $[\widehat{\rho}_2, p_i]$, (ii) V^J is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$, (iii) the constraint set of (8) is compact, (iv) if a price p_j is in the constraint set of (8), then all prices $\rho \in [\widehat{\rho}_2, p_j]$ are also in the constraint set. Then we use properties (i)-(iv) to show statements of Lemma 2 are true. Let us first prove (i)-(iv)

Property (i): We begin by noting that V^J is continuous in p_j on $[\widehat{\rho}_2, p_i)$. This is true based on the fact that both components of V^J (V_1 and V_2) are continuous in p_j on $[\widehat{\rho}_2, p_i)$, and the fact that the sum of two continuous functions is a continuous function. We are left to rule out a potential discontinuity at $p_j = p_i$, since both V_1 and V_2 can be discontinuous at this point. A discontinuity in V_1 or V_2 can only exist at $p_j = p_i$ if $x_1 + x_2 > D(p_i, \omega)$. Let us consider this case and compare V^J at $p_j = p_i$ to V^J as $p_j \uparrow p_i$.

For $p_j = p_i$, $V^J = p_i D(p_i, \omega)$. For $p_j < p_i$, $V^J = p_i D(p_i, \omega)\left(1 - x_j / D(p_j, \omega)\right) + p_j x_j$. Now we take the limit as $p_j \uparrow p_i$ and

$$\begin{aligned} \lim_{p_j \uparrow p_i} V^J &= p_i D(p_i, \omega)\left(1 - \frac{x_j}{D(p_i, \omega)}\right) + p_i x_j \\ &= p_i D(p_i, \omega). \end{aligned}$$

Property (ii): If $x_1 + x_2 \le D(p_i, \omega)$, then $V^J = p_1 x_1 + p_2 x_2$, which is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$.

It remains to show that when $x_1 + x_2 > D(p_i, \omega)$, if $p_j > p_j'$, then $V^J > V'^J$. We write out $V^J > V'^J$ below

$$p_i D(p_i, \omega)\left(1 - \frac{x_j}{D(p_j, \omega)}\right) + p_j x_j > p_i D(p_i, \omega)\left(1 - \frac{x_j}{D(p_j', \omega)}\right) + p_j' x_j. \tag{13}$$

After some routine algebra, the expression in (13) is

$$x_j \cdot \left[(p_j - p_j') + \frac{p_i D(p_i, \omega)}{D(p_j, \omega) \cdot D(p_j', \omega)}\left(D(p_j', \omega) - D(p_j, \omega)\right)\right] > 0. \tag{14}$$

By definition $p_j > p_j'$, which makes the first term in the block parentheses of (14) positive. Since demand is strictly decreasing in price, $p_j > p_j'$ implies that $D(p_j', \omega) > D(p_j, \omega)$, which makes the second term inside block parentheses of (14) positive. Putting the afore mentioned properties together the inequality in (14) is indeed verified.

Property (iii): Now we show the constraint set is compact. First note that since both additively separable pieces of $G_i^a(p, x, \omega)$ is continuous in p_j on $[\widehat{\rho}_2, p_i]$, the function $G_i^a(p, x, \omega)$

inherits the continuity. The constraint $G_i^a(p, x, \omega) \leq \mathcal{L}_i$ is a weak inequality composed continuous functions in p_j on on $[\widehat{\rho}_2, p_i]$, hence it is closed. The constraint set is a closed subset of the compact set $[\widehat{\rho}_2, p_i]$, hence it is compact.

Property (iv): The final property is that if a price p_j is in the constraint set of (8), then all prices $\rho \in [\widehat{\rho}_2, p_j]$ are also in the constraint set. We show this by showing that $G_i^a(p, x, \omega)$ is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$. This property is almost immediately inspection of the (5), as the first term on the right-hand side is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$ and second term the right-hand side terms (including the negative sign) is weakly increasing in p_j on $[\widehat{\rho}_2, p_i]$. Since $\mathcal{L}_i$ is a constant, showing the maximal gain is strictly increasing is sufficient to prove this property.

Now we turn to the statements of the lemma. First (i) and (iii) are sufficient conditions for the existence of a solution to (8). Second, (ii) and (iv) give us the equilibrium character in the lemma. Simply put, the maximizer $\widetilde{p}_j(p_i)$ must be the largest price in the constraint set; this follows immediately from (ii). Based on (iv), if the price is less than the joint profit maximizing price, then it is uniquely defined by the constraint set holding with equality. ■

Proof of Lemma 3. The proof is organized as follows: First, we prove that a solution exists to (9). This is done by showing that $\widetilde{V}^J(p_i, x, \omega) = V^J(\widetilde{p}_j(p_i), p_i, x, \omega)$ is continuous in p_i on $[\widehat{\rho}_2, \rho^m]$, and that the constraint set is compact. Second, we prove the solution is unique by establishing that $\widetilde{V}^J(p_i, x, \omega)$ is strictly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$.

(*Existence*) Before considering the continuity of $\widetilde{V}^J(p_i, x, \omega)$, we address the properties of $\widetilde{p}_j(p_i)$. Particularly, we show that $\widetilde{p}_j(p_i)$ is continuous and weakly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$. $G_i^a(p, x, \omega)$ is continuous in both prices as a function of both prices it is jointly continuous. Since $\widetilde{p}_j(p_i)$ is defined by $G_i^a(p, x, \omega) = \mathcal{L}_i$ if $\widetilde{p}_j(p_i) < \rho^m$, then the joint continuity of $G_i^a(p, x, \omega$ implies $\widetilde{p}_j(p_i)$ is continuous in p_i on $[\widehat{\rho}_2, \rho^m]$. By inspection of (5), we can see that $G_i^a(p, x, \omega)$ is strictly decreasing in p_i on $[\widehat{\rho}_2, \rho^m]$. From the proof of Lemma 2 property (iv), we know that $G_i^a(p, x, \omega)$ is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$. Hence, to hold the equality $G_i^a(p, x, \omega) = \mathcal{L}_i$, a increase in p_i must be compensated with and increase in p_j. (Note this is only relevant if $p_i < \rho^m$, which implies that $p_j < \rho^m$.)

Now let us turn to showing that $\widetilde{V}^J(p_i, x, \omega)$ is continuous in p_i on $[\widehat{\rho}_2, \rho^m]$. Based on the proof of Lemma 2 property (i), V^J is continuous in both prices individually. A function continuous in two arguments is jointly continuous in both arguments. Hence, V^J is jointly continuous in both prices for p such that $p_i \in [\widehat{\rho}_2, \rho^m]$ and $p_j \in [\widehat{\rho}_2, p_i]$. Combining the two facts we have established that V^J is jointly continuous and $\widetilde{p}_j(p_i)$ is continuous, it must be that $\widetilde{V}^J(p_i, x, \omega)$ is continuous in p_i on $[\widehat{\rho}_2, \rho^m]$.

In order to establish that the constraint set is compact, we show that the function $\widetilde{g}_j^a(p_i, x, \omega) = g_j^a(\widetilde{p}_j(p_i), p_i, x, \omega)$ is continuous in p_i on $[\widehat{\rho}_2, \rho^m]$. Again we appeal to the fact that this function is continuous in both prices individually to imply that it is jointly continuous in those prices. Based on this fact and that $\widetilde{p}_j(p_i)$ is continuous, $\widetilde{g}_j^a(p_i, x, \omega)$ must be continuous in p_i on $[\widehat{\rho}_2, \rho^m]$. The constraint $\widetilde{g}_j^a(p_i, x, \omega) \leq \mathcal{L}_j$ is a weak inequality composed of continuous functions in p_j on $[\widehat{\rho}_2, p_i]$, hence it is closed. The constraint set is a closed subset of the compact set $[\widehat{\rho}_2, p_i]$, hence it is compact.

Based the fact that the objective function is continuous and the constraint set is compact, a maximum exists.

(*Character*): Next we move to establishing the character of $\widetilde{p}_i^{ai}$.

The first step is establishing that $\widetilde{V}^J(p_i, x, \omega)$ is strictly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$. Based of the proof of Lemma 2 property (ii), V^J is strictly increasing in p_j on $[\widehat{\rho}_2, p_i]$. We have also established that $\widetilde{p}_j(p_i)$ is weakly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$. We are left to show that V^J is increasing in p_i on $[p_j, \rho^m]$. We take the case that $x_1 + x_2 > D(p_i, \omega)$ and show that if $p_i > p_i'$, then $V^J > V'^J$, which is written out below

$$\begin{aligned} p_i D(p_i, \omega)\left(1 - \frac{x_j}{D(p_j, \omega)}\right) + p_j x_j &> p_i' D(p_i', \omega)\left(1 - \frac{x_j}{D(p_j, \omega)}\right) + p_j x_j. \\ p_i D(p_i, \omega) &> p_i' D(p_i', \omega). \end{aligned}$$

The final inequality must hold based on the fact that revenue is strictly increasing on $[0, \rho^m]$. Recall that the expected revenue is strictly increasing on $[0, \rho^m]$ because $\rho D(\rho, \omega)$ is strictly concave and ρ^m its unique maximizer. Since $\widetilde{V}^J(p_i, x, \omega)$ is strictly increasing in p_i on $[\widehat{\rho}_2, \rho^m]$, $\widetilde{p}_i^{ai}$ is the largest price in the compact constraint set. Uniqueness of $\widetilde{p}_i^{ai}$ immediately follows from the fact that the a compact subset of $\mathbb{R}$ has unique maximal element.

Now let us move to showing that $\widetilde{p}_i^{ai}$ is such that $\widetilde{g}_j^a(\widetilde{p}_i^{ai}, x, \omega) = \mathcal{L}_j$ or $\widetilde{p}_i^{ai} = \rho^m$. Suppose to the contrary that $\widetilde{p}_i^{ai} < \rho^m$ and $\widetilde{g}_j^a(\widetilde{p}_i^{ai}, x, \omega) < \mathcal{L}_j$. We proceed by showing a contradiction; that there exists $\eta > 0$, such that $\rho = \widetilde{p}_i^{ai} + \eta < \rho^m$ and $\widetilde{g}_j^a(\rho, x, \omega) \leq \mathcal{L}_j$.

Since $\mathcal{L}_j > 0$, there exists $\epsilon > 0$ such that $\widetilde{g}_j^a(\widetilde{p}_i^{ai}, x, \omega) = \mathcal{L}_j - \epsilon$. We write out the gain:

$$\begin{aligned} \widetilde{g}_j^a(\rho, x, \omega) &= \left(\widetilde{p}_i^{ai} + \eta\right) \min\{D(\widetilde{p}_i^{ai} + \eta, \omega), x_j\} - p_j x_j \\ &\leq \left(\widetilde{p}_i^{ai} + \eta\right) \min\{D(\widetilde{p}_i^{ai}, \omega), x_j\} - p_j x_j \\ &= \widetilde{g}_j^a(\widetilde{p}_i^{ai}, x, \omega) + \eta \min\{D(\widetilde{p}_i^{ai}, \omega), x_j\}. \end{aligned}$$

Define $\eta > 0$ such that $\eta \leq \epsilon / \min\{D(\widetilde{p}_i^{ai}, \omega), x_j\}$ and notice that $\widetilde{g}_j^a(\rho, x, \omega) = \mathcal{L}_j$. Hence, $\rho > \widetilde{p}_i^{ai}$ is in the constraint set, which combined with the fact that $\widetilde{V}^J(p_i, x, \omega)$ strictly increasing in p_i on $[p_j, \rho^m]$ is a contradiction to the optimality of $\widetilde{p}_i^{ai}$. ■

Proof Lemma 4. The first part of the statement is essentially Fabra (2006) Proposition 1. The second part is just, for all x, define the term

$$\underline{\delta}(x) = \sup\left\{\delta \in [0,1] \,|\forall \delta' \leq \delta,\, \Delta(x) \setminus \boldsymbol{\phi}^n = \varnothing\right\}.$$

and notice that it meets the criteria of the statement of the proposition. ■

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