Appendix: Proofs for "The Quantity Flexibility Contract and Supplier-Customer Incentives"

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The following Lemma summarizes results that will be used in the proofs that follow.

LEMMA 1. *Since $G(r|\mu)$ (defined in (1)) is a newsvendor expected profit:*
(a) $G'(r|\mu) = (p+s-c) - (p+s-u)\Gamma(r-\mu)$
(b) since $G(r|\mu)$ is unimodal, with z_ϵ defined such that $\mu + z_\epsilon\sigma_\epsilon = argmax_r\, G(r|\mu)$,

$$sgn[G'(r|\mu)] = \begin{cases} + & \text{when } r < \mu + z_\epsilon\sigma_\epsilon \\ 0 & \text{when } r = \mu + z_\epsilon\sigma_\epsilon \\ - & \text{when } r > \mu + z_\epsilon\sigma_\epsilon \end{cases}$$

(c) since $G(r|\mu)$ is concave, $G''(r|\mu) < 0$ for all r. ■

PROOF OF PROPOSITION 1. By definition, $Q^*_{CC} = F^{-1}((p+s-m)/(p+s-u))$. Also, $Q^*_{NC} = A + B$ where $A = \Theta^{-1}((c-m)/(c-u))$ and $B = \Gamma^{-1}((p+s-c)/(p+s-u))$. (When ϵ is assumed to be normal, $\Gamma^{-1}((p+s-c)/(p+s-u)) = z_\epsilon\sigma_\epsilon$. The following proof does not require this assumption, hence we use the more general representation for the B term).

By basic algebraic logic we can infer that for arbitrary μ, ϵ, A, and B:

$$\{\mu + \epsilon < A + B\} \quad \Rightarrow \quad \{(\mu < A) \text{ OR } (\epsilon < B)\} \tag{5}$$

This can be established most convincingly by noting that the contrapositive of the statement, i.e., $\{(\mu \geq A) \text{ AND } (\epsilon \geq B)\} \Rightarrow \{\mu + \epsilon \geq A + B\}$, is obviously true. Equation (5) implies

$$Pr\{(\mu < A) \;\text{ OR }\; (\epsilon < B)\} > Pr\{\mu + \epsilon < A + B\} = F(Q^*_{NC}) \tag{6}$$

since the set of outcomes described by the left hand side of (5) must be contained within the set on the right hand side. But

$$Pr\{(\mu < A) \;\text{ OR }\; (\epsilon < B)\} = Pr\{\mu < A\} + Pr\{\epsilon < B\} - Pr\{\mu < A\} \cdot Pr\{\epsilon < B\}$$

$$= \left(\frac{c-m}{c-u}\right) + \left(\frac{p+s-c}{p+s-u}\right) - \left(\frac{c-m}{c-u}\right)\left(\frac{p+s-c}{p+s-u}\right) = \frac{p+s-m}{p+s-u} = F(Q^*_{CC}) \tag{7}$$

where the first equality is the "Addition Law" from elementary probability, also invoking the independence of μ and ϵ. Equations (6) and (7) together show that $F(Q^*_{CC}) > F(Q^*_{NC})$, which establishes that $Q^*_{CC} > Q^*_{NC}$. Hence, total system profit is suboptimal under the *NC* control system, since production other than Q^*_{CC} has resulted[14]. ■

PROOF OF PROPOSITION 2. $\frac{d\pi_{EM,QF}}{dQ} = (c-m) - (c-u)\Theta(Q - z_\epsilon\sigma_\epsilon)$ and $\frac{d\pi^2_{EM,QF}}{dQ^2} = -(c-u)\Theta'(Q - z_\epsilon\sigma_\epsilon)$. The sign of $\frac{d\pi^2_{EM,QF}}{dQ^2}$ verifies the concavity of the objective to be maximized, so we proceed to the Kuhn-Tucker conditions (cf. Rockafellar 1970, Theorem 28.3). These indicate that it will be sufficient to find $\{Q^*, \lambda\}$ such that $\lambda \geq 0$, $Q^* \geq q_{QF}(1+\alpha)$, $(c-m) - (c-u)\Theta(Q^* - z_\epsilon\sigma_\epsilon) + \lambda = 0$ and $\lambda \cdot [q_{QF}(1+\alpha) - Q^*] = 0$. There are two possible solutions:

(i) $\lambda = 0$, which suggests $Q^* = \Theta^{-1}\left(\frac{c-m}{c-u}\right) + z_\epsilon\sigma_\epsilon = Q^*_{NC} \geq q_{QF}(1+\alpha)$

(ii) $\lambda > 0$, which means that $Q^* = q_{QF}(1+\alpha) = \Theta^{-1}\left(\frac{c-m+\lambda}{c-u}\right) + z_\epsilon\sigma_\epsilon > Q^*_{NC}$

These may be combined as $Q^*_{QF}(q_{QF}) = max\{Q^*_{NC}, q_{QF}(1+\alpha)\}$. So an EM willing to exceed the mandatory $q_{QF}(1+\alpha)$ will build all the way to Q^*_{NC}, its preference absent the contract.

The retailer's procedure for choosing q_{QF} indicates that the EM strictly exceeds the obligatory production only when $q_{QF} = 0$. Note that $Q^*_{QF}(q_{QF}) > q_{QF}(1+\alpha)$ only when $Q^*_{NC} > q_{QF}(1+\alpha)$, which occurs on the range $0 \leq q_{QF} < \frac{Q^*_{NC}}{1+\alpha}$. But on this range, the retailer's objective function is clearly strictly decreasing in q_{QF} since this entails a reduction in purchase commitment without any sacrifice of production availability. Hence $q_{QF} = 0$ will result, meaning that the QF contract imposes no constraint on the retailer at all. (This could also be established by formal analysis of the retailer's optimization problem.). In such a case, the only difference to the EM between the QF and *NC* settings is an additional constraint in the former. The EM therefore would not offer the QF contract unless it could anticipate (based on the problem parameters) that the retailer would respond with $q_{QF} > 0$. ■

PROOF OF PROPOSITION 3. The retailer's initial forecast is $q^*_{QF} \equiv argmax_q\{\pi_{R,QF}\left(r^*_{QF}\left(q, Q^*_{QF}(q), \mu\right)\right)\}$, where $\pi_{R,QF}\left(r^*_{QF}\left(q, Q^*_{QF}(q), \mu\right)\right) = E_\mu\{G\left(r^*_{QF}\left(q, Q^*_{QF}(q), \mu\right)|\mu\right)\}$ and $r^*_{QF}\left(q, Q^*_{QF}(q), \mu\right) = (\mu + z_\epsilon\sigma_\epsilon) \perp [q(1-\omega), q(1+\alpha)]$. Parametrizing only on q for clarity, the objective function may be written explicitly as

$$\pi_{R,QF}(q) = \int_{\mu+z_\epsilon\sigma_\epsilon \leq q(1-\omega)} G(q(1-\omega)|\mu)d\Theta(\mu) \tag{8}$$

$$+ \int_{q(1-\omega)\leq\mu+z_\epsilon\sigma_\epsilon\leq q(1+\alpha)} G((\mu+z_\epsilon\sigma_\epsilon)|\mu)d\Theta(\mu) + \int_{\mu+z_\epsilon\sigma_\epsilon\geq q(1+\alpha)} G(q(1+\alpha)|\mu)d\Theta(\mu)$$

[14]In previous versions of this paper, Proposition 1 required that μ be normally distributed. The above generalization was suggested by Rhonda Righter.

By Leibniz' Rule (cf. Ross 1980) and Lemma 1(b),

$$\frac{d\pi_{R,QF}(q)}{dq} = (1-\omega)\int_{\mu+z_\epsilon\sigma_\epsilon\le q(1-\omega)} G'(q(1-\omega)|\mu)d\Theta(\mu) \tag{9}$$
$$+ (1+\alpha)\int_{\mu+z_\epsilon\sigma_\epsilon\ge q(1+\alpha)} G'(q(1+\alpha)|\mu)d\Theta(\mu)$$

$$\frac{d\pi^2_{R,QF}(q)}{dq^2} = (1-\omega)^2\int_{\mu+z_\epsilon\sigma_\epsilon\le q(1-\omega)} G''(q(1-\omega)|\mu)d\Theta(\mu)$$
$$+ (1+\alpha)^2\int_{\mu+z_\epsilon\sigma_\epsilon\ge q(1+\alpha)} G''(q(1+\alpha)|\mu)d\Theta(\mu)$$

By Lemma 1(c), $\frac{d\pi^2_{R,QF}(q)}{dq^2} < 0$, hence (3). Existence of a positive solution to (3) can be established as follows. From the unimodality of G, when $q = 0, G' > 0$ so that $\frac{d\pi_{R,QF}(0)}{dq} > 0$, and $q = \infty$ implies $G' < 0$ so that $\frac{d\pi_{R,QF}(\infty)}{dq} < 0$. By the continuity of $\frac{d\pi_{R,QF}(q)}{dq}$ and the Intermediate Value Theorem (Thm. 18.2 of Ross 1980), a positive root to $\frac{d\pi_{R,QF}(q)}{dq} = 0$ exists.

Part (b) is obvious on applying the change of variable $Q = q(1+\alpha)$ to (3):

$$\int_{\mu+z_\epsilon\sigma_\epsilon\le\frac{Q}{\psi}} G'\left(\frac{Q}{\psi}|\mu\right)d\Theta(\mu) + \psi\int_{\mu+z_\epsilon\sigma_\epsilon\ge Q} G'(Q|\mu)d\Theta(\mu) = 0 \tag{10}$$

This substitution has the connotation of ceding the retailer direct control over the production decision Q, which is a natural implication of Proposition 2.

Comparative statics can be obtained most compactly by using Q^*_{QF} as the decision and ψ as the measure of flexibility; extrapolation to q^*_{QF}, α and ω is straightforward. By implicitly differentiating (10),

$$\frac{dQ^*_{QF}}{dc} = \frac{-\psi\left[\Theta\left(\frac{Q^*_{QF}}{\psi} - z_\epsilon\sigma_\epsilon\right) + \psi\left[1 - \Theta\left(Q^*_{QF} - z_\epsilon\sigma_\epsilon\right)\right]\right]}{(p+s-u)\left[\int_{\mu+z_\epsilon\sigma_\epsilon\le\frac{Q^*_{QF}}{\psi}}\Gamma'\left(\frac{Q^*_{QF}}{\psi} - \mu\right)d\Theta(\mu) + \psi^2\int_{\mu+z_\epsilon\sigma_\epsilon\ge Q^*_{QF}}\Gamma'\left(Q^*_{QF} - \mu\right)d\Theta(\mu)\right]}$$

$$\frac{dQ^*_{QF}}{d\psi} = \frac{(p+s-u)Q^*_{QF}\int_{\mu+z_\epsilon\sigma_\epsilon\le\frac{Q^*_{QF}}{\psi}}\Gamma'\left(\frac{Q^*_{QF}}{\psi} - \mu\right)d\Theta(\mu) + \psi^2\int_{\mu+z_\epsilon\sigma_\epsilon\ge Q^*_{QF}} G'\left(Q^*_{QF}\middle|\mu\right)d\Theta(\mu)}{(p+s-u)\psi\left[\int_{\mu+z_\epsilon\sigma_\epsilon\le\frac{Q^*_{QF}}{\psi}}\Gamma'\left(\frac{Q^*_{QF}}{\psi} - \mu\right)d\Theta(\mu) + \psi^2\int_{\mu+z_\epsilon\sigma_\epsilon\ge Q^*_{QF}}\Gamma'\left(Q^*_{QF} - \mu\right)d\Theta(\mu)\right]}$$

$\frac{dQ^*_{QF}}{dc}$ is clearly negative, and $\frac{dQ^*_{QF}}{d\psi}$ is positive because the second integral in the numerator positive by Lemma 1(b). $\frac{dQ^*_{QF}}{d\alpha}$ and $\frac{dQ^*_{QF}}{d\omega}$ inherit the sign of $\frac{dQ^*_{QF}}{d\psi}$ since ψ increases in both α and

ω. Since $q^*_{QF} = Q^*_{QF}/(1+\alpha)$, $\frac{dq^*_{QF}}{dc} = \frac{dQ^*_{QF}}{dc}\frac{1}{1+\alpha} > 0$ and $\frac{dq^*_{QF}}{d\omega} = \frac{dQ^*_{QF}}{d\omega}\frac{1}{1+\alpha} > 0$, while $\frac{dq^*_{QF}}{d\alpha} = \frac{dQ^*_{QF}}{d\alpha}\frac{1}{1+\alpha} - \frac{Q^*_{QF}}{(1+\alpha)^2}$ is indeterminate.

While $\frac{d\pi^*_{R,QF}}{dc}$ can be explicitly quantified by direct differentiation and much algebra, it is inconvenient to analyze. However, its sign may be obtained from the following line of reasoning. We denote $\pi^*_{R,QF}$ as $\pi_{R,QF}\big(Q^*_{QF}(c);c\big)$ just for clarity in this proof. For any $c' > c$,

$$\pi_{R,QF}\big(Q^*_{QF}(c);c\big) > \pi_{R,QF}\big(Q^*_{QF}(c');c\big)$$

since $Q^*_{QF}(c)$ maximizes the retailer's expected profit for a given c. Next, compare $\pi_{R,QF}\big(Q^*_{QF}(c');c\big)$ and $\pi_{R,QF}\big(Q^*_{QF}(c');c'\big)$. After observing μ, the retailer faces a newsvendor problem with fixed purchase constraints. Since the common Q^*_{QF} indicates identical constraints in the two settings, the higher procurement cost entails lower optimal profit. Since this holds for every realization of μ, it must remain true after unconditioning on μ to obtain $\pi^*_{R,QF}$. Hence

$$\pi_{R,QF}\big(Q^*_{QF}(c');c\big) > \pi_{R,QF}\big(Q^*_{QF}(c');c'\big)$$

Therefore $c' > c$ implies $\pi_{R,QF}\big(Q^*_{QF}(c);c\big) > \pi_{R,QF}\big(Q^*_{QF}(c');c'\big)$, i.e. $\frac{d\pi^*_{R,QF}}{dc} < 0$.

For $\frac{d\pi^*_{R,QF}}{d\psi}$, express $\pi^*_{R,QF}$ as $\pi_{R,QF}\big(Q^*_{QF}(\psi);\psi\big)$. Then

$$\frac{d\pi^*_{R,QF}}{d\psi} = \frac{d\pi_{R,QF}(Q;\psi)}{d\psi}\Big|_{Q=Q^*_{QF}(\psi)} = -\frac{Q}{\psi^2}\int_{\mu+z_\epsilon\sigma_\epsilon\le\frac{Q}{\psi}} G'\left(\frac{Q}{\psi}\right)d\Theta(\mu)\Big|_{Q=Q^*_{QF}(\psi)} > 0$$

The first equality is due to the "Envelope Theorem" (cf. Varian 1984)[15]. The sign is due to Lemma 1(b), and the signs of $\frac{d\pi^*_{R,QF}}{d\alpha}$ and $\frac{d\pi^*_{R,QF}}{d\omega}$ follow immediately.

The numerical example of §8 is sufficient to show the non-monotonicity of $\pi^*_{EM,QF}$. ■

PROOF OF PROPOSITION 4. When $\omega = 1$, (9) is strictly positive by Lemma 1(b). Hence, the retailer prefers as large a q_{QF} as possible, so as to maximize the likelihood of obtaining his desired quantity after observing μ. This implies mathematically an infinite forecast which in turn makes the EM's required production, $Q^*_{QF} = q^*_{QF}(1+\alpha)$, arbitrarily large. Naturally, this exceeds the production quantity that results from central-control.

When $(\alpha,\omega) = (0,0)$, the retailer's problem reverts to standard newsvendor structure since the forecast is tantamount to a concrete purchase (so $q^*_{QF} = Q^*_{QF}$). As this decision is made without seeing μ, $F()$ is the appropriate distribution to use. In fact, the retailer's problem is

[15] In general the total derivative is

$$\frac{d\pi^*_{R,QF}}{d\psi} = \frac{d\pi_{R,QF}(Q;\psi)}{d\psi}\Big|_{Q=Q^*_{QF}(\psi)} + \frac{d\pi_{R,QF}\big(Q^*_{QF}(\psi);\psi\big)}{dQ^*_{QF}(\psi)}\frac{dQ^*_{QF}(\psi)}{d\psi}$$

$\frac{d\pi_{R,QF}(Q^*_{QF}(\psi);\psi)}{dQ^*_{QF}(\psi)}$ vanishes by the definition of $Q^*_{QF}(\psi)$.

analogous to the central-control problem (see §4), except the procurement cost is $c > m$. Hence $Q^*_{QF} = F^{-1}((p+s-c)/(p+s-u)) < F^{-1}((p+s-m)/(p+s-u)) = Q^*_{CC}$. ■

PROOF OF PROPOSITION 5. Proposition 4(b) states that for arbitrary $c > m$, $Q^*_{QF}(c, \psi(0,0)) < Q^*_{CC}$. But the proof of Proposition 4(a) showed that for any c, as ω→1 (i.e. $\psi(0,\omega)$→∞), $Q^*_{QF}(c, \psi(0,\omega))$→∞ (monotonicity comes from Proposition 3). Since $Q^*_{QF}()$ is continuous in ψ, for any given c there must exist a unique ψ_c such that $Q^*_{QF}(c, \psi_c) = Q^*_{CC}$.

We invert the relationship in (a) to denote the c associated with a given ψ as $\overline{c}(\psi)$. This relationship is defined implicitly by (10) on setting Q to the desired production quantity (here the target is Q^*_{CC}, although this technique works for any other). We abbreviate (3) as $A(\overline{c}, \psi) + \psi B(\overline{c}) = 0$, where $A(c, \psi) = \int_{\mu + z_\epsilon \sigma_\epsilon \leq \frac{Q^*_{CC}}{\psi}} G'\left(\frac{Q^*_{CC}}{\psi}|\mu\right) d\Theta(\mu)$ and $B(c) = \int_{\mu + z_\epsilon \sigma_\epsilon \geq Q^*_{CC}} G'(Q^*_{CC}|\mu) d\Theta(\mu)$, and differentiate implicitly to obtain

$$\frac{d\overline{c}}{d\psi} = -\left[\frac{B(\overline{c}) + \frac{\partial A}{\partial \psi}}{\frac{\partial A}{\partial \overline{c}} + \psi \frac{dB}{d\overline{c}}}\right]$$

Lemma 1(b) indicates $B(\overline{c}) > 0$, and $\frac{\partial A}{\partial \psi} = -\frac{Q^*_{CC}}{\psi} \int_{\mu + z_\epsilon \sigma_\epsilon \leq \frac{Q^*_{CC}}{\psi}} G''\left(\frac{Q^*_{CC}}{\psi}|\mu\right) d\Theta(\mu) > 0$ by Lemma 1(c). Finally, $\frac{\partial A}{\partial \overline{c}} = \int_{\mu + z_\epsilon \sigma_\epsilon \leq \frac{Q^*_{CC}}{\psi}} \frac{dG'\left(\frac{Q^*_{CC}}{\psi}|\mu\right)}{d\overline{c}} d\Theta(\mu) = -\Theta\left(\frac{Q^*_{CC}}{\psi} - z_\epsilon \sigma_\epsilon\right) < 0$ and $\frac{dB}{d\overline{c}} = \int_{\mu + z_\epsilon \sigma_\epsilon \geq Q^*_{CC}} \frac{dG'(Q^*_{CC}|\mu)}{d\overline{c}} d\Theta(\mu) = -[1 - \Theta(Q^*_{CC} - z_\epsilon \sigma_\epsilon)] < 0$, where the second equality in each makes use of Lemma 1(a). Hence $\frac{d\overline{c}}{d\psi} > 0$. ■

PROOF OF PROPOSITION 6. When $\sigma_\epsilon = 0$, $F() = \Theta()$ and $G(r|\mu) = (p+s-c)r - s\mu - (p+s-u)[r-\mu]^+$, so that (10) simplifies to $[1 - F(Q)](p+s-c) = \frac{1}{\psi} F\left(\frac{Q}{\psi}\right)(c-u)$. It is tedious but straightforward to verify that when the transfer price is $\overline{c}$ as defined in part (a), $Q^*_{QF} = F^{-1}((p+s-m)/(p+s-u)) = Q^*_{CC}$. Also, except for the extreme parameter settings $\psi = 1$ and $\psi = \infty$ ($(\alpha, \omega) = (0,0)$ and $\omega = 1$, respectively) that were explored in §5.2, $\overline{c} \in (m, p+s)$.

The proof of (b) and (c) examines the retailer's profit over ψ's domain of $[1, \infty)$. As ψ is lowered to 1, $\overline{c}$ drops to m. In this limiting case the retailer has no flexibility but receives the product at the EM's production cost, hence will achieve the optimal central-control profit. At the other extreme, as $\psi \to \infty$, $\overline{c}$ approaches $(p+s)$, in which case the retailer makes no profit. The retailer's expected profit under an efficient QF contract is continuous and can be shown to be strictly decreasing in ψ by direct differentiation (Lariviere 1999 details this). Thus, any desired portion of the profit can be shifted to the EM by increasing ψ. Since efficiency is not guaranteed when $\sigma_\epsilon > 0$, this logic does not extend to the general model. ■