Online supplement to manuscript:

Supply chain strategies and international tax arbitrage

1. Note on asymmetric taxation of profits and losses

In addition to the notation introduced in the paper, we use the following additional notation in this subsection:

Table 1 Additional notation

θ_B	Number of years allowed by tax law for carryback of tax credits
θ_F	Number of years allowed by tax law for carryforward of tax credits
z	Interest rate
V_j	Random variable for profit at time j with mean m and standard deviation s

This paper considers a single period of activity. However, in practice, tax credits from a current period can also be applied to either earlier or future periods. The following analysis will evaluate the impact of making this explicit.

Let θ_B represent the length of the carryback window (currently 2 years in the U.S.) and θ_F the length of the carryforward window (currently 20 years in the U.S.). Limitations on the lengths of these windows, the presence of an interest rate z, and the possibility that profits will not be enough to use up any accumulated tax credits combine to create divergence between models of symmetric and asymmetric taxation. We show next that this divergence is insignificant in many real settings.

Let V_0 represent before-tax profit in period 0. $V_0^+ = \max\{0, V_0\}$ and $V_0^- = -\min\{0, V_0\}$, such that tV_0^- represents the tax credit in period 0. The expected exercised value of tV_0^- estimates how to "discount" the value of tax credit. If the interest rate is non-zero, tax credit should be used as soon as possible. A naive alternative, which we assess for tractable analysis, is to spread the tax credit across the entire $(\theta_B + \theta_F)$-year time window. The expected exercised value of tV_0^- is then:

$$\mathbb{E}[\text{Exercised value of } tV_0^-] = tV_0^- \frac{1}{\theta_B + \theta_F} \left(\sum_{j=1}^{\theta_B} (1+z)^j + \sum_{j=1}^{\theta_F} \frac{1}{(1+z)^j} \right). \tag{1}$$

Assuming the current time windows for companies operating in the U.S. ($\theta_B = 2$ and $\theta_F = 20$), the expected exercised value of tax credit under the naive strategy equals 0.71 $(0.77, 0.84, 0.91)$ of tV_0^- when the interest rate is 4% $(3\%, 2\%, 1\%)$. Provided that the firm realizes profit soon after seeing a loss, the optimal practice of using the tax credit as soon as possible should result in an even smaller gap. Moreover, since tax credits get netted with tax liability, the MNF can consider the NPV of the average income as follows: $NPV = \frac{1}{\theta_B+1+\theta_F} \left(\sum_{j=1}^{\theta_B} (1+z)^j V_{-j} + V_0 + \sum_{j=1}^{\theta_F} \frac{1}{(1+z)^j V_j} \right)$. Assuming the V_j are i.i.d. normal random variables with mean m and standard deviation s, the average income is also normally distributed with mean m and standard deviation $\frac{s}{\theta_B+1+\theta_F}$, making the probability of a negative observation very small. We thus conclude that treating the tax credits and tax liabilities symmetrically is a reasonable approximation provided that the risk-free interest rate z is low and the time window to use tax credits is large.

2. Derivation of the division manager's Certainty Equivalent

$$U(\alpha,\beta,\gamma_i,e,q,\varepsilon) = -Exp[-r(W_i(\alpha,\beta,\gamma_i,e,q,\varepsilon)-\frac{ke^2}{2})]$$
$$f(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}}Exp[-\frac{\varepsilon^2}{2\sigma^2}]$$

The expectation of the utility function with respect to ε is:

$$\mathbb{E}[U(\alpha,\beta,\gamma_i,e,q,\varepsilon)] = -\int_{-\infty}^{\infty} Exp[-r(\alpha+\beta\gamma_i\pi(e,q,\varepsilon)-\frac{ke^2}{2})]f(\varepsilon)d\varepsilon =$$

$$\mathbb{E}[U(\alpha,\beta,\gamma_i,e,q,\varepsilon)] = -\int_{-\infty}^{\infty} Exp[-r\alpha-r\beta\gamma_i q(1+e-q-c+\varepsilon)+\frac{rke^2}{2}]f(\varepsilon)d\varepsilon$$

Only one part of this integral depends on ε. That part can be simplified as follows:

$$-\int_{-\infty}^{\infty} Exp[-r\beta\gamma_i q\varepsilon]f(\varepsilon)d\varepsilon =$$

$$-\int_{-\infty}^{\infty} Exp[-r\beta\gamma_i q\varepsilon]\frac{1}{\sigma\sqrt{2\pi}}Exp[-\frac{\varepsilon^2}{2\sigma^2}]d\varepsilon =$$

$$-\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}Exp[-r\beta\gamma_i q\varepsilon-\frac{\varepsilon^2}{2\sigma^2}]d\varepsilon =$$

$$-\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}Exp[-\frac{1}{2\sigma^2}(2r\beta\gamma_i q\varepsilon\sigma^2+\varepsilon^2+r^2\beta^2\gamma_i^2q^2\sigma^4)+\frac{r^2\beta^2\gamma_i^2q^2\sigma^2}{2}]d\varepsilon =$$

$$-\int_{-\infty}^{\infty} Exp[\frac{r^2\beta^2\gamma_i^2q^2\sigma^2}{2}]d\varepsilon \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}Exp[-\frac{1}{2\sigma^2}(\varepsilon+r\beta\gamma_i q\varepsilon\sigma^2)^2]}_{\substack{\text{Normal density with mean} \\ -r\beta\gamma_i q\varepsilon\sigma^2 \text{ and variance } \sigma^2}} d\varepsilon = -Exp[\frac{r^2\beta^2\gamma_i^2q^2\sigma^2}{2}]$$

Then the expected utility can be written as:
$\mathbb{E}[U(\alpha,\beta,\gamma_i,e,q,\varepsilon)] = -Exp[-rCE]$, where $CE = \alpha+\beta\gamma_i\pi(e,q,0)-\frac{1}{2}r\beta^2\gamma_i^2q^2\sigma^2-\frac{ke^2}{2}$.

3. Sensitivity Analysis of the solutions to Problems N, C, L, and F

3.1. Problem N

The following table summarizes the direction of change of the optimal solution to Problem N (No Distributor).

Measure	Effort (e_N^*)	Quantity (q_N^*)	Price (p_N^*)	Profit (Π_N^*)
Cost of effort premium (δ)	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Cost of effort (k)	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Product cost (c)	$\downarrow$	$\downarrow$	$\downarrow\uparrow^{\dagger}$	$\downarrow$
HQ tax rate (t)	$-$	$-$	$-$	$\downarrow$
Fixed distributor's cost (d)	$-$	$-$	$-$	$\downarrow$

First derivatives of all decisions and the resulting profit for Problem N are summarized below. Since $\delta k > \frac{1}{2}$ and $c < 1$, the signs of the derivatives are straight-forward:

j:	$\frac{de_N^*}{dj}$	$\frac{dq_N^*}{dj}$	$\frac{dp_N^*}{dj}$	$\frac{d\Pi_N^*}{dj}$
Cost of effort (k)	$-\frac{2(1-c)}{(1-2k)^2} \leq 0$	$-\frac{1-c}{(1-2k)^2} \leq 0$	$-\frac{1-c}{(1-2\delta k)^2} \leq 0$	$-\frac{(1-c)^2 B(0)}{2(1-2\delta k)^2} \leq 0$
Product cost (c)	$\frac{1}{1-2\delta k} \leq 0$	$\frac{\delta k}{1-2\delta k} \leq 0$	$\frac{\delta k-1}{2\delta k-1} \begin{cases} \leq 0 & \forall \delta k \in [1/2, 1] \\ > 0 & \forall \delta k > 1 \end{cases}$	$\frac{(c-1)\delta k B(0)}{2\delta k-1} \leq 0$
HQ tax rate (t)	$-$	$-$	$-$	$-\frac{(c-1)^2 \delta k}{4\delta k-2} \leq 0$
Fixed distributor's cost (d)	$-$	$-$	$-$	$-(1-t)$

The results on sensitivity of effort, quantity, and price on production cost c could be generalized as follows:

Let $c_e(e)$ represent the cost of effort, which is convex increasing in e and let $\pi(e,q,\epsilon) = q\,(p(e,q,\epsilon)-c) = q\,(\hat{p}(e,q)+\epsilon-c)$, where $p(e,q,\epsilon) = \hat{p}(e,q)+\epsilon$ is the inverse demand curve with additive noise, $\hat{p}(e,q)$ is increasing in marketing effort e and decreasing in q, and $R(e,q,) = q\,(\hat{p}(e,q))$ is jointly concave in e and q and continuously differentiable, implying that the Hessian matrix is semidefinite: $a_{11} \leq 0$, $a_{22} \leq 0$, and $a_{11}a_{22} - a_{12}^2 \geq 0$, where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \triangleq \begin{pmatrix} \frac{\partial^2 R(e,q)}{\partial e^2} & \frac{\partial^2 R(e,q)}{\partial e \partial q} \\ \frac{\partial^2 R(e,q)}{\partial q \partial e} & \frac{\partial^2 R(e,q)}{\partial q^2} \end{pmatrix}.$$

Assuming a non-ill behaved function $R(e,q)$, $a_{12} = a_{21} = a_0$.

The No Distributor's optimization problem can then be written as:

$$\max_{e,q} \pi_N(e,q) = \mathbb{E}_\epsilon\left[q\,(\hat{p}(e,q)+\epsilon-c) - \delta c_e(e) - d\right](1-t)$$

$\pi_N(e,q)$ is jointly concave in e and q by assumptions on $R(e,q)$ and $c_e(e)$. Hence, optimal e^* and q^* solve the first order conditions:

$$\begin{cases} \frac{\partial \pi_N(e,q)}{\partial e} = \frac{\partial R(e,q)}{\partial e} - \delta \frac{dc_e(e)}{de} = \frac{q\partial \hat{p}(e,q)}{\partial e} - \delta \frac{dc_e(e)}{de} = 0 \\ \frac{\partial \pi_N(e,q)}{\partial q} = \frac{\partial R(e,q)}{\partial q} - c = \hat{p}(e,q) + q\frac{q\partial \hat{p}(e,q)}{\partial q} - c = 0. \end{cases} \tag{2}$$

We differentiate the first order conditions in (**??**) above wrt c and apply the Envelope Theorem to the profit function of c (other sensitivities can be obtained in the similar way):

$$\begin{cases} \frac{q\partial^2 \hat{p}(e,q)}{\partial e^2}\frac{de}{dc} - \delta \frac{d^2 c_e(e)}{de^2}\frac{de}{dc} + \frac{\partial^2 R(e,q)}{\partial e \partial q}\frac{dq}{dc} = 0 \\ \frac{\partial^2 R(e,q)}{\partial e \partial q}\frac{de}{dc} + \frac{\partial^2 R(e,q)}{\partial q^2}\frac{dq}{dc} - 1 = 0. \end{cases}$$

Replacing $\delta \frac{d^2 c_e(e)}{de^2}$ with b, we can re-write as:

$$\begin{cases} (a_{11}-b)\frac{de}{dc}+a_0\frac{dq}{dc}=0 \\ a_0\frac{de}{dc}+a_{22}\frac{dq}{dc}-1=0 \end{cases}$$

We can evaluate two cases: $a_0=0$ and $a_0 \neq 0$.

Case $a_0=0$: From $(a_{11}-b)\frac{de}{dc}=0$, we have $\frac{de}{dc}=0$ if $a_{11}\neq b$.

And $\frac{dq}{dc}=\frac{1}{a_{22}}<0$ if $a_{22}\neq 0$.

$a_0=0$ implies that the quantity and effort do not have a joint effect on the revenue, which is rarely practical.

Case $a_0\neq 0$: From the second equation, we have $\frac{de}{dc}=\frac{1}{a_0}\left(1-a_{22}\frac{dq}{dc}\right)$. Plugging into first, we get $(a_{11}-b)\frac{1}{a_0}\left(1-a_{22}\frac{dq}{dc}\right)+a_0\frac{dq}{dc}=0$ and after some algebra, we can obtain: $\frac{dq}{dc}=\frac{b-a_{11}}{a_0^2-a_{22}a_{11}+a_{22}b}$ and $\frac{de}{dc}=\frac{a_0}{a_0^2-a_{22}a_{11}+a_{22}b}$. Recalling that $b>0$, $a_{11}\leq 0$, $a_{22}\leq 0$, and $a_{11}a_{22}-a_{12}^2\geq 0$, we conclude that $\frac{dq}{dc}\leq 0$ and quantity decrease in c.

The sign of $\frac{de}{dc}$ depends on a_0:

If $a_0>0$ (i.e., q and e are complementary in the revenue function $R(e,q)$), the effort is *decreasing* in c. Since $\hat{p}(e,q)$ is increasing in marketing effort e and decreasing in q, price is *nonmonotone* in c.

If $a_0<0$ (i.e. q and e are substitutable in the revenue function $R(e,q)$), the effort is *increasing* in c.

3.2. Problem C

The following table summarizes the direction of change of the optimal solution to Problem C (Commissionnaire).

Measure	Effort (e_C^*)	Quantity (q_C^*)	Price (p_C^*)	Profit (Π_C^*)
Cost of effort (k)	↓	↓	↓	↓
Product cost (c)	↓	↓	↓↑	↓
HQ tax rate (t)	–	–	–	↓
Foreign tax rate (τ)	–	–	–	↓↑†
Profit allocation (γ_C)	–	–	–	↑
Reservation wage (w)	–	–	–	↓

First derivatives of decisions are the same as in Problem N and are omitted. Comparative statics of HQ's profit in Problem C are summarized below. All the signs follow from the following conditions: $k>\frac{1}{2}$, $c<1$, $1>t>\tau$, and $\gamma_C<1$.

j:	$\frac{d\Pi_C^*}{dj}$
Cost of effort (k)	$-\frac{(1-c)^2 B(\gamma_C)}{2(1-2k)^2} \leq 0$
Product cost (c)	$\frac{(c-1)kB(\gamma_C)}{2k-1} \leq 0$
HQ tax rate (t)	$-\frac{(c-1)^2 k(1-\gamma_C)}{4k-2} \leq 0$
Foreign tax rate (τ)	$w - \frac{(c-1)^2 k\gamma_C}{4k-2} \begin{cases} \leq 0 & \forall w \leq \frac{(c-1)^2 k\gamma_C}{4k-2} \\ > 0 & \forall w > \frac{(c-1)^2 k\gamma_C}{4k-2} \end{cases}$
Profit allocation (γ_C)	$\frac{(c-1)^2 k(t-\tau)}{4k-2} \geq 0$
Reservation wage (w)	$-(1-\tau) \leq 0$

The results on sensitivity of effort, quantity, and price on production cost c could be generalized similarly to the results in Problem N. The only difference compared to Problem N is as follows: in Problem C, HQ decides α, e, and q to maximize her expected profit such that $\alpha \geq w$. However, since the constraint is binding at optimality, $\alpha^* = w$, sensitivity analysis for Problem C becomes exactly the same.

3.3. Problem L

The following table summarizes the direction of change of the optimal solution to Problem L (Limited Risk Distributor).

Measure	Effort (e_L^*)	Quantity (q_L^*)	Price (p_L^*)	Bonus (β_L^*)	Profit (Π_L^*)
Cost of effort (k)	↓	↓	↓	↓	↓
Risk exposure (ξ)	↓	↓	↓	↓	↓
Product cost (c)	↓	↓	↓↑	–	↓
HQ tax rate (t)	↓	↓	↓	↓	↓
Foreign tax rate (τ)	↑	↑	↑	↑	↓↑
Profit allocation (γ_L)	↑	↑	↑	↓	↑
Reservation wage (w)	–	–	–	–	↓

Signs of derivatives follow from the modeling assumptions. For derivatives that can be $\gtreqless 0$, we find the threshold at which each derivative changes sign by setting it equal to 0.

j:	$\frac{de_L^*}{dj}$	$\frac{dq_L^*}{dj}$
Cost of effort (k)	$\frac{-2(1-c)(1-\tau)(2k\xi+1)B(\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$	$-\frac{(1-c)(1-\tau)(2k\xi+1)B(\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$
Risk exposure (ξ)	$\frac{-2k^2(1-c)(1-\tau)B(\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$	$-\frac{k^2(1-c)(1-\tau)B(\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$
Product cost (c)	$\frac{-B(\gamma_L)}{2Ak-B(\gamma_L)} \leq 0$	$\frac{-kA}{(2Ak-B(\gamma_L))} \leq 0$
HQ tax rate (t)	$\frac{-2kA(1-c)k(1-\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$	$\frac{-k(1-c)A(1-\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$
Foreign tax rate (τ)	$\frac{2k(1-c)(k\xi+1)(1-t)(1-\gamma_L)}{(2Ak-B(\gamma_L))^2} \geq 0$	$\frac{k(1-c)(k\xi+1)(1-t)(1-\gamma_L)}{(2Ak-B(\gamma_L))^2} \geq 0$
Profit allocation (γ_L)	$\frac{2k(1-c)A(t-\tau)}{(2Ak-B(\gamma_L))^2} \geq 0$	$\frac{k(1-c)A(t-\tau)}{(2Ak-B(\gamma_L))^2} \geq 0$
Reservation wage (w)	–	–

j:	$\frac{dp_L^*}{dj}$	$\frac{d\beta_L^*}{dj}$	$\frac{d\Pi_L^*}{dj}$
Cost of effort (k)	$-\frac{(1-c)(1-\tau)(2k\xi+1)B(\gamma_L)}{(B(\gamma_L)-2kA)^2} \leq 0$	$\frac{-\xi B(\gamma_L)}{(k\xi+1)A\gamma_L} \leq 0$	$\frac{-(1-c)^2(1-\tau)(2k\xi+1)B(\gamma_L)^2}{2(2Ak-B(\gamma_L))^2} \leq 0$
Risk exposure (ξ)	$-\frac{k^2(1-c)(1-\tau)B(\gamma_L)}{(B(\gamma_L)-2kA)^2} \leq 0$	$\frac{-kB(\gamma_L)}{(k\xi+1)A\gamma_L} \leq 0$	$\frac{-k^2(1-c)^2(1-\tau)B(\gamma_L)^2}{2(2Ak-B(\gamma_L))^2} \leq 0$
Product cost (c)	$\frac{Ak-B(\gamma_L)}{2Ak-B(\gamma_L)}$	$-$	$\frac{-Ak(1-c)B(\gamma_L)}{2Ak-B(\gamma_L)} \leq 0$
HQ tax rate (t)	$\frac{-k(1-c)A(1-\gamma_L)}{(B(\gamma_L)-2kA)^2} \leq 0$	$-\frac{1-\gamma_L}{A\gamma_L} \leq 0$	$\frac{-A^2k^2(1-c)^2(1-\gamma_L)}{(2Ak-B(\gamma_L))^2} \leq 0$
Foreign tax rate (τ)	$\frac{k(1-c)(1-t)(k\xi+1)(1-\gamma_L)}{(B(\gamma_L)-2kA)^2} \geq 0$	$\frac{(1-t)(1-\gamma_L)}{(1-\tau)A\gamma_L} \geq 0$	$w-\frac{(k\xi+1)k(1-c)^2(2kA\gamma_L(1-\tau)-B(\gamma_L)^2)}{2(2Ak-B(\gamma_L))^2}$
Profit allocation (γ_L)	$\frac{k(1-c)A(t-\tau)}{(B(\gamma_L)-2kA)^2} \geq 0$	$-\frac{1-t}{A\gamma_L^2} \leq 0$	$\frac{k^2(1-c)^2(1-\tau)^2(k\xi+1)^2(t-\tau)}{(2Ak-B(\gamma_L))^2} \geq 0$
Reservation wage (w)	$-$	$-$	$-(1-\tau)$

3.4. Problem F

For Problem F, we present relevant comparative statics in the proof of Proposition 6 below.

4. Proof of Propositions

Proof of Proposition 1 For the profit function $\Pi_N(e,q) = \left(\mathbb{E}_\varepsilon[\pi(e,q,\varepsilon)] - \frac{\delta k e^2}{2}\right)(1-t)$ the Hessian matrix is: $\begin{pmatrix} -\delta k(1-t) & 1-t \\ 1-t & -2(1-t) \end{pmatrix}$. The second-order principal minor is $(2\delta k-1)(1-t)^2$, which is positive since $\delta k > \frac{1}{2}$. The function is thus jointly concave in e and q. First-order conditions ($\frac{\partial \Pi_N(e,q)}{\partial e} = (1-t)(q-e\delta k) = 0$ and $\frac{\partial \Pi_N(e,q)}{\partial q} = (1-t)(1-c+e-2q)$) then yield $e_N^* = \frac{1-c}{2\delta k-1}$ and $q_N^* = \frac{\delta k(1-c)}{2\delta k-1}$.

Proof of Proposition 2 Omitted due to similarity to Proof of Proposition 1.

Proof of Proposition 3 Proposition 3 identifies the threshold $\hat{\gamma}_C$ that makes $\Pi_C = \Pi_N$. This follows from solving $\frac{\delta k(1-c)^2}{2}\frac{B(0)}{(2\delta k-1)} - d(1-t) = \frac{k(1-c)^2}{2}\frac{B(\gamma_C)}{2k-1} - w(1-\tau)$ to find $\hat{\gamma}_C = \frac{2(2k-1)(w(1-\tau)-d(1-t))}{(1-c)^2k(t-\tau)} - \frac{(\delta-1)(1-t)}{(2\delta k-1)(t-\tau)}$.

Comparative statics of $\hat{\gamma}_C$:

1. With respect to w: $\frac{d\hat{\gamma}_C}{dw} = \frac{2(2k-1)(1-\tau)}{(1-c)^2k(t-\tau)} > 0$;
2. With respect to d: $\frac{d\hat{\gamma}_C}{dd} = -\frac{2(2k-1)(1-t)}{(1-c)^2k(t-\tau)} < 0$;
3. Since $0<c<1$, $(1-c)^2$ decreases in c, $\hat{\gamma}_C$ increases in c.
4. With respect to t: $\frac{d\hat{\gamma}_C}{dt} = \frac{(\delta-1)(1-\tau)}{(2\delta k-1)(t-\tau)^2} - \frac{2(2k-1)(d(\tau-1)+w(1-\tau))}{(c-1)^2k(t-\tau)^2} = \frac{1-\tau}{(t-\tau)^2}\left(\frac{2(2k-1)(d-w)(2\delta k-1)+(\delta-1)k(1-c)^2}{k(1-c)^2(2\delta k-1)}\right)$. Hence, when $w < d + \frac{(\delta-1)k(1-c)^2}{2(2k-1)(2\delta k-1)}$, $\hat{\gamma}_C$ increases in t, and decreases otherwise.
5. With respect to δ: $\frac{d\hat{\gamma}_C}{d\delta} = -\frac{(2k-1)(1-t)}{(2\delta k-1)^2(t-\tau)} < 0$;
6. With respect to k: $\frac{d\hat{\gamma}_C}{dk} = \frac{2(w(1-\tau)-d(1-t))}{(1-c)^2k^2(t-\tau)} - \frac{2(1-\delta)\delta(1-t)}{(2\delta k-1)^2(t-\tau)}$;

Hence, when $w > d\frac{1-t}{1-\tau} + \frac{\delta(1-\delta)(1-t)(1-c)^2k^2}{(2\delta k-1)^2(1-\tau)}$, $\hat{\gamma}_C$ increases in k, and decreases otherwise.

7. With respect to τ: $\frac{d\hat{\gamma}_C}{d\tau} = -\frac{1-t}{(t-\tau)^2}\left(\frac{2(2k-1)(d-w)}{(1-c)^2k} + \frac{\delta-1}{2\delta k-1}\right)$, hence $\hat{\gamma}_C$ decreases in τ when $w < d + \frac{(\delta-1)(1-c)^2k}{2(2k-1)(2\delta k-1)}$, and increases otherwise.

Proof of Proposition 4 We first find the best-response behavior of the manager. The manager's certainty equivalent CE is concave in e ($\frac{\partial^2 CE(e)}{\partial e^2} = -k$) and the first-order condition identifies the optimal effort level e_L: $\frac{\partial CE(e)}{\partial e} = \beta q \gamma_L - ek = 0$, hence $e_L = \frac{\beta q \gamma_L}{k}$.

We substitute this into the profit function to obtain $\overline{\Pi}_L(\beta, q)$ and then optimize over β and q. We optimize sequentially. The firm's profit function is concave in β: $\frac{\partial^2 \overline{\Pi}_L(\beta,q)}{\partial \beta^2} = -\frac{q^2 A \gamma_L^2}{k} < 0$. We find the extreme point for β using the first-order condition ($\frac{\partial \overline{\Pi}_L(\beta,q)}{\partial \beta} = 0$) and discover it to be independent of q: $\beta_L^* = \frac{B(\gamma_L)}{A\gamma_L}$. We then substitute β_L^* into the objective function and find the optimal q_L^*. $\overline{\Pi}_L(\beta_L^*, q)$ is concave in q: $\frac{\partial^2 \overline{\Pi}_L(\beta_L^*,q)}{\partial q^2} = \frac{B(\gamma_L)(B(\gamma_L)-2Ak)}{Ak} < 0$ since $k > \frac{1}{2}$ and $A > B(\gamma_L)$. The first-order condition then delivers the optimal q:$\frac{\partial \overline{\Pi}_L(\beta_L^*,q)}{\partial q} = \frac{B(\gamma_L)(qB(\gamma_L)-Ak(c+2q-1))}{Ak} = 0$ and hence, $q_L^* = \frac{A(1-c)k}{2Ak - B(\gamma_L)}$.

Proof of Proposition 5 We first find the threshold $\hat{\gamma}_L$ that makes $\Pi_L^* = \Pi_C^*$:

$$\frac{k(1-c)^2}{2}\frac{B(\gamma_C)}{2k-1} - w(1-\tau) = \frac{k(1-c)^2}{2}\frac{B(\hat{\gamma}_L)}{2k - \frac{B(\hat{\gamma}_L)}{A}} - w(1-\tau)$$

This simplifies to:

$$\begin{aligned}
\frac{B(\gamma_C)}{B(\hat{\gamma}_L)} &= \frac{2k-1}{2k - \frac{B(\hat{\gamma}_L)}{A}} \\
2kB(\gamma_C) - \frac{B(\hat{\gamma}_L)}{A}B(\gamma_C) &= (2k-1)B(\hat{\gamma}_L) \\
B(\hat{\gamma}_L) &= \frac{2AkB(\gamma_C)}{2Ak - A + B(\gamma_C)} = \frac{2kB(\gamma_C)}{2k - 1 + \frac{B(\gamma_C)}{A}}
\end{aligned}$$

We can now express $\hat{\gamma}_L$ as $\hat{\gamma}_L = \frac{\frac{2kB(\gamma_C)}{2k-1+\frac{B(\gamma_C)}{A}}}{t-\tau} - \frac{1-t}{t-\tau} = \frac{1}{t-\tau}\frac{2kB(\gamma_C)}{2k-1+\frac{B(\gamma_C)}{A}} - \frac{1-t}{t-\tau}$.

First derivatives show comparative statics of $\hat{\gamma}_L$.

1. Effect of tax rate t (we use the chain rule). For ease of exposition, we replace $B(\gamma_C)$ with B in the following derivation.

$$\begin{aligned}
\frac{\partial \hat{\gamma}_L(B,t)}{\partial t} &= \frac{-2AkB + (A(2k-1)+B)(1-\tau)}{(t-\tau)^2 (A(2k-1)+B)} \\
\frac{\partial \hat{\gamma}_L}{\partial B} &= \frac{2kA^2(2k-1)}{(t-\tau)(A(2k-1)+B)^2} \\
\frac{dB}{dt} &= \gamma_C - 1
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\gamma}_L}{dt} &= \frac{\partial \hat{\gamma}_L}{\partial B}\frac{dB}{dt} + \frac{\partial \hat{\gamma}_L}{\partial t} = \\
&= \frac{2kA^2(2k-1)(\gamma_C - 1)}{(t-\tau)(A(2k-1)+B)^2} + \frac{-2AkB + (A(2k-1)+B)(1-\tau)}{(t-\tau)^2 (A(2k-1)+B)} = \\
&= \frac{2kA^2(2k-1)(\gamma_C-1)(t-\tau) - 2AkB\,(A(2k-1)+B) + (A(2k-1)+B)^2(1-\tau)}{(t-\tau)^2 \left(A(2k-1)+B\right)^2}
\end{aligned}$$

$$= \frac{A^2(2k-1)(\tau-1) - 2AB(k(B+2\tau-2)-\tau+1) - B^2(\tau-1)}{(t-\tau)^2(A(2k-1)+B)^2}$$
$$= \frac{((A-B)^2+4ABk)(1-\tau) - 2Ak(A(1-\tau)+B^2)}{(t-\tau)^2(A(2k-1)+B)^2}$$
$$= \frac{(1-\tau)\left(-(2k-1)(A-B)^2 - 2B^2k^2\xi\right)}{(t-\tau)^2(A(2k-1)+B)^2} < 0$$

Hence, $\hat{\gamma}_L$ decreases in t.

2. Effect of cost of effort (k):

$$\frac{d\hat{\gamma}_L}{dk} = \frac{2Bk\xi(1-\tau)}{(t-\tau)(B+(2k-1)A)} + \frac{2BA}{(t-\tau)(B+(2k-1)A)} - \frac{2BkA((2k-1)\xi(1-\tau)+2A)}{(t-\tau)(B+(2k-1)A)^2}$$

$$\frac{d\hat{\gamma}_L}{dk} = \frac{2B(1-\tau)\left((2k\xi+1)B-(1-\tau)(k\xi+1)^2\right)}{(t-\tau)\left(A(2k-1)+B\right)^2} < 0$$

Since $(2k\xi+1) < (k\xi+1)^2$ and $B < (1-\tau)$, $((2k\xi+1)B-(1-\tau)(k\xi+1)^2) < 0$ and $\frac{d\hat{\gamma}_L}{dk} < 0$.

3. Effect of risk exposure (ξ):

$$\frac{d\hat{\gamma}_L}{d\xi} = \frac{2k^2(1-\tau)B\left(\gamma_C\right)^2}{(t-\tau)\left((2k-1)A+B\left(\gamma_C\right)\right)^2} > 0$$

4. Effect of profit allocation percentage (γ_C):

$$\frac{d\hat{\gamma}_L}{d\gamma_C} = \frac{2A^2k(2k-1)}{(A(2k-1)+B\left(\gamma_C\right))^2} > 0$$

5. Effect of foreign tax rate τ:

Taking the full derivative with respect to τ (for clean exposition, we replace $B(\gamma_C)$ with B).

$$\frac{d\hat{\gamma}_L}{d\tau} = -\frac{1-t}{(t-\tau)^2} + \frac{2k\left(B^2(t-\tau)\frac{dA}{d\tau} + (2k-1)A^2\left((t-\tau)\frac{dB}{d\tau}+B\right) + AB^2\right)}{(t-\tau)^2((2k-1)A+B)^2}$$

$$\frac{dA}{d\tau} = -(k\xi+1)$$
$$\frac{dB}{d\tau} = -\gamma_C$$

$(t-\tau)\frac{dB}{d\tau}+B$ simplifies to: $-(t-\tau)\gamma_C + (1-t) + (t-\tau)\gamma_C = 1-t$.

$$\frac{d\hat{\gamma}_L}{d\tau} = -\frac{1-t}{(t-\tau)^2} + \frac{2k\left(-B^2(t-\tau)(k\xi+1) + (2k-1)A^2(1-t) + AB^2\right)}{(t-\tau)^2((2k-1)A+B)^2}$$

$AB^2 - B^2(t-\tau)(k\xi+1)$ simplifies to: $B^2(A-(t-\tau)(k\xi+1)) = B^2(k\xi+1)(1-\tau-t+\tau) = B^2(k\xi+1)(1-t)$.

$$
\begin{aligned}
\frac{d\hat{\gamma}_L}{d\tau} &= -\frac{1-t}{(t-\tau)^2} + \frac{2k\left(B^2(k\xi+1)(1-t)+(2k-1)A^2(1-t)\right)}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= -\frac{1-t}{(t-\tau)^2} + \frac{2k(1-t)\left(B^2(k\xi+1)+(2k-1)A^2\right)}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= \frac{-(1-t)((2k-1)A+B)^2 + 2k(1-t)\left(B^2(k\xi+1)+(2k-1)A^2\right)}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= -(1-t)\frac{((2k-1)A+B)^2 - 2kB^2(k\xi+1) - 2k(2k-1)A^2}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= -(1-t)\frac{((2k-1)A)^2 + 2(2k-1)AB + B^2 - 2kB^2 - 2k(2k-1)A^2 - 2k^2B^2\xi}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= -(1-t)\frac{(2k-1)(2k-1)A^2 - 2k(2k-1)A^2 + 2(2k-1)AB - B^2(2k-1) - 2k^2B^2\xi}{(t-\tau)^2((2k-1)A+B)^2} = \\
&= -(1-t)\frac{-(2k-1)(A-B)^2 - 2k^2B^2\xi}{(t-\tau)^2((2k-1)A+B)^2} = \frac{(1-t)\left((2k-1)(A-B)^2 + 2B^2k^2\xi\right)}{(t-\tau)^2(A(2k-1)+B)^2} > 0
\end{aligned}
$$

Hence, $\hat{\gamma}_L$ increases in τ.

Proof of Proposition 6 First, find the optimal response of the manager. The certainty equivalent for Problem F:

$$
\begin{aligned}
CE_F(\alpha,\beta,e,q) &= \alpha - \frac{ke^2}{2} + \beta\gamma_F\left(\pi(e,q,0)\right) - \frac{1}{2}rq^2\gamma_F^2\beta^2\sigma^2 \qquad (3)\\
&= \alpha - \frac{ke^2}{2} + \beta\gamma_F q\left(1+e-q-c\right) - \frac{1}{2}\xi q^2\gamma_F^2\beta^2. \qquad (4)
\end{aligned}
$$

We first show that $CE_F(\alpha,\beta,e,q)$ is jointly concave in e and q.

$$
\begin{aligned}
&\frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial^2 e} = -k < 0;\\
&\frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial^2 q} = -\beta\gamma_F(\beta\gamma_F+2) < 0;\\
&\frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial e\partial q} = \beta\gamma_F;\\
&\frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial^2 e} * \frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial^2 q} - \left(\frac{\partial^2 CE_F(\alpha,\beta,e,q)}{\partial e\partial q}\right)^2 = \beta^2\gamma_F^2(k\xi-1) + 2k\beta\gamma_F > 0.
\end{aligned}
$$

In other words, the Hessian matrix is $\begin{pmatrix} -k & \beta\gamma_F \\ \beta\gamma_F & -\beta\gamma_F\left(\beta\xi\gamma_F+2\right)\end{pmatrix}$.

The second principal minor is then $\beta\gamma_F\left(\beta\gamma_F(k\xi-1)+2k\right) > \beta\gamma_F\left(\beta\gamma_F(k\xi-1)+1\right) = \beta\gamma_F\left(\beta\gamma_F k\xi + 1 - \beta\gamma_F\right) \geq 0$.

The function is thus jointly concave in e and q.

First-order conditions then can deliver the manager's best-response q and e:

$$
\begin{aligned}
\frac{\partial CE_F(\alpha,\beta,e,q)}{\partial q} &= \beta\gamma_F\left(-c+e-\beta\xi q\gamma_F - 2q + 1\right) = 0\\
\frac{\partial CE_F(\alpha,\beta,e,q)}{\partial e} &= \beta q\gamma_F - ek = 0
\end{aligned}
$$

Solving for e and q:

$$\begin{aligned}
&q = \frac{ek}{\beta\gamma_F} \\
&\beta\gamma_F\left(-c + e - \beta\xi\frac{ek}{\beta\gamma_F}\gamma_F - 2\frac{ek}{\beta\gamma_F} + 1\right) = 0 \\
&-c\beta\gamma_F + e\beta\gamma_F - \beta\xi ek\gamma_F - 2ek + \beta\gamma_F = 0 \\
&e_F = \frac{\beta\gamma_F(c-1)}{(\beta\gamma_F - \beta\xi k\gamma_F - 2k)} = \frac{\beta\gamma_F(1-c)}{(\beta\gamma_F(k\xi-1)+2k)} \\
&q_F = \frac{e_F k}{\beta\gamma_F} = \frac{k(1-c)}{(\beta\gamma_F(k\xi-1)+2k)}
\end{aligned}$$

Substitute $e_F = \frac{\beta(1-c)\gamma_F}{(\beta\gamma_F(k\xi-1)+2k)}$ and $q_F = \frac{k(1-c)}{(\beta\gamma_F(k\xi-1)+2k)}$ into $\Pi_F(\alpha_F, \beta, e_F, q_F)$:

$$\begin{aligned}
\overline{\Pi}_F(\beta) = \Pi_F(\alpha_F, \beta, e_F, q_F) &= q_F(1 + \frac{\beta q_F\gamma_F}{k} - q_F - c)B(\gamma_F) - \frac{1}{2}\frac{\beta^2 q_F^2\gamma_F^2}{k}A - w(1-\tau) \\
&= q_F(1-c)B(\gamma_F) + q_F^2\left((\frac{\beta\gamma_F}{k} - 1)B(\gamma_F) - \frac{1}{2}\frac{\beta^2\gamma_F^2}{k}A\right) - w(1-\tau) \\
&= \frac{k(1-c)^2 B(\gamma_F)}{(\beta\gamma_F(k\xi-1)+2k)} + q_F^2\left((\frac{\beta\gamma_F}{k} - 1)B(\gamma_F) - \frac{\beta^2\gamma_F^2}{2k}A\right) - w(1-\tau) \\
&= \frac{k(1-c)^2 B(\gamma_F)}{(\beta\gamma_F(k\xi-1)+2k)} + \frac{k^2(1-c)^2}{(\beta\gamma_F(k\xi-1)+2k)^2}\left((\frac{\beta\gamma_F}{k} - 1)B(\gamma_F) - \frac{\beta^2\gamma_F^2}{2k}A\right) - w(1-\tau) \\
&= \frac{k(1-c)^2}{(\beta\gamma_F(k\xi-1)+2k)}\left(B(\gamma_F) + \frac{k\left((\frac{\beta\gamma_F}{k} - 1)B(\gamma_F) - \frac{\beta^2\gamma_F^2}{2k}A\right)}{(\beta\gamma_F(k\xi-1)+2k)}\right) - w(1-\tau) \\
&= \frac{k(1-c)^2\left(2B\left(\gamma_F\right)\left(\beta k\xi\gamma_F + k\right) - A\beta^2\gamma_F^2\right)}{2\left(\beta\gamma_F(k\xi-1)+2k\right)^2} - w(1-\tau).
\end{aligned}$$

Substituting into the MNF's profit function yields:

$$\overline{\Pi}_F(\beta) = \frac{k(c-1)^2\left(2B\left(\gamma_F\right)\left(\beta k\xi\gamma_F + k\right) - A\beta^2\gamma_F^2\right)}{2\left(\beta\gamma_F(k\xi-1)+2k\right)^2} - w(1-\tau).$$

Look at the part that is relevant for the derivative (we replace $B\left(\gamma_F\right)$ with B for cleaner exposition in the derivation):

$$\frac{2B(\beta k\xi\gamma_F + k) - A\beta^2\gamma_F^2}{\left(\beta\gamma_F(k\xi-1)+2k\right)^2}$$

Numerator:

$$2B(\beta k\xi\gamma_F + k) - A\beta^2\gamma_F^2$$

Derivative of the numerator with respect to β:

$$2Bk\xi\gamma_F - 2A\beta\gamma_F^2 = 2\gamma_F(Bk\xi - A\beta\gamma_F)$$

Denominator:

$$(\beta\gamma_F(k\xi-1)+2k)^2 \triangleq Z^2$$

Derivative of the denominator with respect to β:

$$2\left(\beta\gamma_F(k\xi-1)+2k\right)\gamma_F(k\xi-1) \stackrel{\triangle}{==} 2Z\gamma_F(k\xi-1)$$

Numerator of $\frac{d\frac{2B(\beta k\xi\gamma_F+k)-A\beta^2\gamma_F^2}{(\beta\gamma_F(k\xi-1)+2k)^2}}{d\beta}$:

$$(2Bk\xi\gamma_F-2A\beta\gamma_F^2)Z^2-2Z\gamma_F(k\xi-1)(2B(\beta k\xi\gamma_F+k)-A\beta^2\gamma_F^2)=$$
$$Z\left((2Bk\xi\gamma_F-2A\beta\gamma_F^2)\left(\beta\gamma_F(k\xi-1)+2k\right)-2\gamma_F(k\xi-1)(2B\left(\beta k\xi\gamma_F+k\right)-A\beta^2\gamma_F^2)\right)$$

Let $Y=\gamma_F(k\xi-1)$

$$Z\left(2\gamma_F(Bk\xi-A\beta\gamma_F)\left(\beta Y+2k\right)-2Y(2Bk\left(\beta\xi\gamma_F+1\right)-A\beta^2\gamma_F^2)\right)=$$
$$Z\left((2\gamma_F Bk\xi-2\gamma_F^2A\beta)\left(\beta Y+2k\right)-4YBk(\beta\xi\gamma_F+1)+2YA\beta^2\gamma_F^2\right)=$$
$$Z\left((2\gamma_F Bk\xi-2\gamma_F^2A\beta)\left(\beta Y+2k\right)-4YBk\beta\xi\gamma_F-4YBk-+2YA\beta^2\gamma_F^2\right)=$$
$$Z\left(2\gamma_F Bk\xi\beta Y-2\gamma_F^2A\beta^2Y+4\gamma_F Bk^2\xi-4k\gamma_F^2A\beta-4YBk\beta\xi\gamma_F-4YBk+2YA\beta^2\gamma_F^2\right)=$$
$$Z\left(2\gamma_F Bk\xi\beta Y+4\gamma_F Bk^2\xi-4k\gamma_F^2A\beta-4YBk\beta\xi\gamma_F-4YBk\right)=$$
$$Z\left(-2\gamma_F Bk\xi\beta Y+4\gamma_F Bk^2\xi-4k\gamma_F^2A\beta-4YBk\right)=$$
$$-2kZ\left(\gamma_F B\xi\beta Y-2\gamma_F Bk\xi+2\gamma_F^2A\beta+2YB\right)=$$
$$-2kZ\left(\gamma_F B\xi\beta Y-2\gamma_F Bk\xi+2\gamma_F^2A\beta+2\gamma_F(k\xi-1)B\right)=$$
$$-2kZ\left(\gamma_F B\xi\beta Y-2\gamma_F Bk\xi+2\gamma_F^2A\beta+2\gamma_F Bk\xi-2\gamma_F B\right)=$$
$$-2kZ\left(\gamma_F B\xi\beta Y+2\gamma_F^2A\beta-2\gamma_F B\right)=$$
$$-2kZ\gamma_F\left(B\xi\beta Y+2\gamma_F A\beta-2B\right)$$

Substituting Y and Z back, we get:

$$-2k\left(\beta\gamma_F(k\xi-1)+2k\right)\gamma_F\left(2\gamma_F A\beta+B(\xi\beta\gamma_F(k\xi-1)-2)\right);$$

Putting together with the remainder of the function (replace B back with $B\left(\gamma_F\right)$):

$$\frac{\partial\overline{\Pi}_F(e_F,q_F)}{\partial\beta}=-\frac{k^2(c-1)^2\gamma_F\left(2A\beta\gamma_F+B\left(\gamma_F\right)\left(\beta\xi\gamma_F(k\xi-1)-2\right)\right)}{\left(\beta\gamma_F(k\xi-1)+2k\right)^3}.$$

Since the denominator of $\frac{\partial \overline{\Pi}_F(e_F,q_F)}{\partial \beta}$ is positive, the numerator is linear in β, and $\frac{\partial \overline{\Pi}_F(\beta)}{\partial \beta}\Big|_{\beta=0} = \frac{(1-c)^2\gamma_F B(\gamma_F)}{4k} > 0$, the first-order condition provides the optimal β:

$$\beta_F^* = \frac{2B(\gamma_F)}{\gamma_F(2A + \xi(k\xi - 1)B(\gamma_F))}$$

.

Substitute β_F^* into the profit function:

$$\overline{\Pi}_F(\beta_F^*) = \frac{k(1-c)^2}{2}\frac{\left(2B\left(\frac{2B}{\gamma_F(2A+\xi(k\xi-1)B)}k\xi\gamma_F + k\right) - A\frac{4B^2}{\gamma_F^2(2A+\xi(k\xi-1)B)^2}\gamma_F^2\right)}{\left(\frac{2B}{\gamma_F(2A+\xi(k\xi-1)B)}\gamma_F(k\xi-1)+2k\right)^2} - (1-\tau)w.$$

Simplifying to:

$$\overline{\Pi}_F(\beta_F^*) = \frac{(1-c)^2k}{4}\frac{(2A + B(\gamma_F)k\xi^2)}{2Ak + B(\gamma_F)(k^2\xi^2-1)}B(\gamma_F) - (1-\tau)w.$$

Comparative statics of q_F^*.

Substituting β_F^* into q_F^*, we get:

$$q_F^*(\beta_F^*) = \frac{(1-c)k}{2\left(\frac{B(k\xi-1)}{2A+B\xi(k\xi-1)} + k\right)}.$$

To evaluate derivatives with respect to k and ξ, we expand A:

$$q_F^*(\beta_F^*) = \frac{(1-c)k}{2\left(\frac{B(k\xi-1)}{2(k\xi+1)(1-\tau)+B\xi(k\xi-1)} + k\right)}.$$

1. Take derivative wrt k:

$$\frac{dq_F^*(\beta_F^*)}{dk} = \frac{B(1-c)\left(B\xi(k\xi-1)^2 + 2(1-\tau)\left(k^2\xi^2 - 2k\xi - 1\right)\right)}{2(k\xi+1)^2(B(k\xi-1)+2k(1-\tau))^2}$$

The second term in the numerator may change sign. Hence, we set it equal to zero to find the threshold.

$$\left(B\xi(k\xi-1)^2 + 2(1-\tau)\left(k^2\xi^2 - 2k\xi - 1\right)\right) = 0$$

Solving for k, we get:

$$k = \frac{B\xi^2 + 2\xi(1-\tau) \pm 2\xi\sqrt{(1-\tau)(B\xi + 2(1-\tau))}}{\xi^2(B\xi + 2(1-\tau))} = \frac{1}{\xi} \pm \frac{2\sqrt{(1-\tau)(B\xi+2(1-\tau))}}{\xi(B\xi + 2(1-\tau))}$$

$$= \frac{1}{\xi} \pm \frac{2\sqrt{(1-\tau)}}{\xi\sqrt{(B\xi + 2(1-\tau))}}$$

Re-write this in terms of $k\xi$:

$$k\xi = 1 \pm \frac{2\sqrt{(1-\tau)}}{\sqrt{(B\xi + 2(1-\tau))}}$$

Since $k\xi > 1$, we only consider one root that is > 1:

$$k\xi = 1 + \frac{2\sqrt{(1-\tau)}}{\sqrt{(B\xi + 2(1-\tau))}}$$

.

When $k\xi > 1 + \frac{2\sqrt{(1-\tau)}}{\sqrt{(B\xi+2(1-\tau))}}$, $q_F^*(\beta_F^*)$ increases in k, and decreases otherwise.

2.

$$\frac{dq_F^*(\beta_F^*)}{d\xi} = \frac{B(1-c)k\left(B(k\xi-1)^2 - 4k(1-\tau)\right)}{2(k\xi+1)^2(B(k\xi-1)+2k(1-\tau))^2}$$

The second term in the numerator may change sign, therefore we set it equal to zero and find the threshold:

$$\left(B(k\xi-1)^2 - 4k(1-\tau)\right) = 0$$

Solving for ξ, we get:

$$\xi = \frac{Bk \pm 2k\sqrt{B(1-\tau)}}{Bk^2} = \frac{1}{k} \pm \frac{2\sqrt{(1-\tau)}}{\sqrt{B}k};$$

Re-write this in terms of $k\xi$:

$$k\xi = 1 \pm \frac{2\sqrt{(1-\tau)}}{\sqrt{B}}$$

.

When $k\xi > 1 + \frac{2\sqrt{(1-\tau)}}{\sqrt{B}}$, $q_F^*(\beta_F^*)$ increases in ξ, and decreases otherwise.

3. To evaluate derivatives with respect to t, τ, and γ_F, we also expand B:

$$q_F^*(\beta_F^*) = \frac{(1-c)k}{2\left(\frac{(1-t+\gamma_F(t-\tau))(k\xi-1)}{2(k\xi+1)(1-\tau)+(1-t+\gamma_F(t-\tau))\xi(k\xi-1)} + k\right)}.$$

Evaluating signs of the derivatives, we find:

$$\frac{dq_F^*(\beta_F^*)}{dt} = \frac{(1-c)(1-\gamma_F)(k\xi-1)(1-\tau)}{(1+k\xi)((B(k\xi-1)+2k(1-\tau))^2)} \geq 0$$

Hence, $q_F^*(\beta_F^*)$ increases t.

$$\frac{dq_F^*(\beta_F^*)}{d\tau} = \frac{-(1-c)(1-\gamma_F)k(k\xi-1)(1-t)}{(1+k\xi)((B(k\xi-1)+2k(1-\tau))^2)} \leq 0$$

Hence, $q_F^*(\beta_F^*)$ decreases in τ.

$$\frac{dq_F^*(\beta_F^*)}{d\gamma_F} = \frac{-(1-c)k(k\xi-1)(1-\tau)(t-\tau)}{(1+k\xi)((B(k\xi-1)+2k(1-\tau))^2)} \leq 0$$

Hence, $q_F^*(\beta_F^*)$ decreases in γ_F.

Behavior of price with respect to c:

$$p_F^* = 1 + e_F^* - q_F^* = \frac{2k + \beta_F^*\gamma_F(k\xi - 1) + \beta_F^*\gamma_F(1-c) - k(1-c)}{2k + \beta_F^*\gamma_F(k\xi - 1)} = \frac{k(1+c) + \beta_F^*\gamma_F(k\xi - c)}{2k + \beta_F^*\gamma_F(k\xi - 1)}.$$

$$\frac{dp_F^*}{dc} = \frac{k - \beta_F^*\gamma_F}{2k + \beta_F^*\gamma_F(k\xi - 1)}$$

.

Since the denominator is positive, $\frac{dp_F^*}{dc} > 0$ when $k - \beta_F^*\gamma_F > 0$:

$$k > \frac{2B}{(2A + \xi(k\xi - 1)B)}$$

$$2Ak + \xi(k\xi - 1)Bk - 2B > 0$$

$$k^2\xi^2 B + k(2A - B\xi) - 2B > 0$$

Hence, $\frac{dp_F^*}{dc} > 0$ when

$$k < \frac{\xi B - 2A - \sqrt{(\xi B - 2A)^2 + 8B^2\xi^2}}{2B\xi^2}$$

or

$$k > \frac{\xi B - 2A + \sqrt{(\xi B - 2A)^2 + 8B^2\xi^2}}{2B\xi^2}.$$

Since $k > \frac{1}{2}$ by Assumption and first root is less than zero, the only relevant condition is:

$$k > \frac{\xi B - 2A + \sqrt{(\xi B - 2A)^2 + 8B^2\xi^2}}{2B\xi^2}.$$

Behavior of profit function with respect to τ:

$$\overline{\Pi}_F(\beta_F^*) = \frac{(1-c)^2 k}{4} \frac{B\left(2A + Bk\xi^2\right)}{2Ak + B\left(k^2\xi^2 - 1\right)} - w(1-\tau).$$

Consider $g(A(\tau), B(\tau)) = \frac{B(\tau)\left(2A(\tau) + B(\tau)k\xi^2\right)}{2A(\tau)k + B(\tau)\left(k^2\xi^2 - 1\right)}$.

$$\frac{dg(A(\tau), B(\tau))}{d\tau} = \frac{\partial g}{\partial A}\frac{dA}{d\tau} + \frac{\partial g}{\partial B}\frac{dB}{d\tau}$$

$$\frac{dB}{d\tau} = -\gamma_F$$

$$\frac{dA}{d\tau} = -(k\xi + 1)$$

$$\frac{\partial g}{\partial A}=\frac{2B(2Ak+B\left(k^2\xi^2-1\right))-2kB(2A+Bk\xi^2)}{(2Ak+B\left(k^2\xi^2-1\right))^2}=\frac{2B^2\left(k^2\xi^2-1\right)-2k^2B^2\xi^2}{(2Ak+B\left(k^2\xi^2-1\right))^2}=-\frac{2B^2}{(2Ak+B\left(k^2\xi^2-1\right))^2}$$

$$\frac{\partial g}{\partial B}=\frac{2(A+Bk\xi^2)(2Ak+B\left(k^2\xi^2-1\right))-B\left(2A+Bk\xi^2\right)\left(k^2\xi^2-1\right)}{(2Ak+B\left(k^2\xi^2-1\right))^2}=$$
$$\frac{4Ak(A+Bk\xi^2)+B^2k\xi^2(k^2\xi^2-1)}{(2Ak+B\left(k^2\xi^2-1\right))^2}.$$

$$\frac{\partial g}{\partial A}\frac{dA}{d\tau}+\frac{\partial g}{\partial B}\frac{dB}{d\tau}=\frac{(k\xi+1)2B^2}{(2Ak+B\left(k^2\xi^2-1\right))^2}+\frac{-\gamma_F(4Ak(A+Bk\xi^2)+B^2k\xi^2(k\xi-1)(k\xi+1))}{(2Ak+B\left(k^2\xi^2-1\right))^2}=$$
$$\frac{(k\xi+1)2B^2-\gamma_F(4Ak(A+Bk\xi^2)+B^2k\xi^2(k\xi-1)(k\xi+1))}{(2Ak+B\left(k^2\xi^2-1\right))^2}=$$
$$\frac{(k\xi+1)B^2(2-\gamma_Fk\xi^2(k\xi-1))-4\gamma_FAk(A+Bk\xi^2)}{(2Ak+B\left(k^2\xi^2-1\right))^2}.$$

$$\frac{d\overline{\Pi}_F}{d\tau}=w-\frac{(1-c)^2k}{4}\frac{4\gamma_FAk(A+Bk\xi^2)-(k\xi+1)B^2(2-\gamma_Fk\xi^2(k\xi-1))}{(2Ak+B\left(k^2\xi^2-1\right))^2}.$$

$$\frac{d\overline{\Pi}_F}{d\tau}=w-\frac{(1-c)^2k}{4}\frac{2B^2-k\gamma_F(B\xi+2(1-\tau))(2A-B\xi(1-k\xi))}{(k\xi+1)(B(1-k\xi)-2k(1-\tau))^2}$$

Hence, $\overline{\Pi}_F(\beta_F^*)$ increases in τ when $w>\frac{(1-c)^2k}{4}\frac{2B^2-k\gamma_F(B\xi+2(1-\tau))(2A-B\xi(1-k\xi))}{(k\xi+1)(B(1-k\xi)-2k(1-\tau))^2}$.

Proof of Proposition 7 First we show that Π_F^* is monotone increasing in γ_F:
$\frac{d\Pi_F^*}{d\gamma_F}=\frac{\partial\Pi_F^*}{\partial B}\frac{dB}{d\gamma_F}=\frac{(1-c)^2k}{4}\frac{4Ak(A+Bk\xi^2)+B^2k\xi^2(k^2\xi^2-1)}{(2Ak+B\left(k^2\xi^2-1\right))^2}(t-\tau)>0.$
Then we find a threshold on γ_F that makes $\Pi_F^*=\Pi_L^*$:

$$\frac{(1-c)^2k}{4}\frac{B(\hat{\gamma}_F)\,(2A+B(\hat{\gamma}_F)k\xi^2)}{2Ak+B(\hat{\gamma}_F)\,(k^2\xi^2-1)}-w(1-\tau)=\frac{(1-c)^2k}{2}\frac{B(\gamma_L)A}{2Ak-B(\gamma_L)}-w(1-\tau);$$

To find $\hat{\gamma}_F$, we need to find $B(\hat{\gamma}_F)$ that satisfies:

$$\frac{B(\hat{\gamma}_F)\,(2A+B(\hat{\gamma}_F)k\xi^2)}{2Ak+B(\hat{\gamma}_F)\,(k^2\xi^2-1)}=\frac{2B(\gamma_L)A}{2Ak-B(\gamma_L)}.$$

Which we rewrite as a quadratic function in $B\left(\hat{\gamma}_F\right)$:

$$k\xi^2B\left(\hat{\gamma}_F\right)^2\left(2Ak-B\left(\gamma_L\right)\right)+2AkB\left(\hat{\gamma}_F\right)\left(2A-k\xi^2B\left(\gamma_L\right)\right)-4A^2kB\left(\gamma_L\right)=0$$

Solving for $B(\hat{\gamma}_F)$:

$$B(\hat{\gamma_F})=\frac{A\left(\sqrt{(2A+k\xi^2B\left(\gamma_L\right))^2-4\xi^2B\left(\gamma_L\right)^2}-2A+k\xi^2B\left(\gamma_L\right)\right)}{\xi^2\left(2Ak-B\left(\gamma_L\right)\right)}$$

Hence, $\hat{\gamma}_F=\frac{\frac{A\left(\sqrt{\left(2A+k\xi^2B(\gamma_L)\right)^2-4\xi^2B(\gamma_L)^2}-2A+k\xi^2B(\gamma_L)\right)}{\xi^2(2Ak-B(\gamma_L))}}{t-\tau}-\frac{1-t}{t-\tau}.$

Proof of Proposition 8 We show the comparison between optimal solutions to all Problems:

1. The ordering of e_C^* and e_L^* follows from the fact that $\beta_L^*\gamma_L<1$ and hence the denominator of e_L^* is larger than the denominator of e_C^* while the numerator of e_L^* is smaller than the denominator of e_C^*.

2. The ordering of q_C^* and q_L^* follows from the fact that $\beta_L^*\gamma_L<1$ and hence the denominator of q_L^* is larger than the denominator of q_C^*.

The ordering of q_L^* and q_F^* follows from the fact that $2k-\beta_L^*\gamma_L<2k+\beta_F^*\gamma_F(k\xi-1)$ and hence the denominator of q_F^* is larger than the denominator of q_L^* while the numerator is the same.

3. Establish the ordering of prices in three steps:

   (a) Show that $p_C^*\geq p_L^*$.

Since $k>\frac{1}{2}$ and $\beta_L^*\gamma_L<1$, the following holds:

$$p_C^*-p_L^*=\frac{(1-c)k\,(1-\beta_L^*\gamma_L)}{(2k-1)\,(2k-\beta_L^*\gamma_L)}\geq 0$$

   (b) Show that $p_C^*\geq p_F^*$.

$$p_C^*-p_F^*=\frac{(c(k-1)+k)(\beta_F^*\gamma_F(k\xi-1)+2k)-(2k-1)(\beta_F^*\gamma_F(k\xi-c)+(c+1)k)}{(2k-1)\,(\beta_F^*\gamma_F(k\xi-1)+2k)}\tag{5}$$

The denominator of **??** is positive, so we examine the numerator:

$$(c(k-1)+k)(\beta_F^*\gamma_F(k\xi-1)+2k)-(2k-1)(\beta_F^*\gamma_F(k\xi-c)+(c+1)k)=$$
$$(c(k-1)+k)\beta_F^*\gamma_F(k\xi-1)+(c(k-1)+k)(2k)-(2k-1)(c+1)k-(2k-1)\beta_F^*\gamma_F(k\xi-c)=$$
$$(c(k-1)+k)\beta_F^*\gamma_F(k\xi-1)+k(1-c)-(2k-1)\beta_F^*\gamma_F(k\xi-c)=$$
$$(1-c)k(\beta_F^*\gamma_F(\xi-k\xi)+1-\beta_F^*\gamma_F)=$$
[Substituting $\beta_F^*\gamma_F=\frac{2B(\gamma_F)}{2A-B(\gamma_F)\xi(1-k\xi)}$:]
$$(1-c)k(\frac{2B(\gamma_F)}{2A-B(\gamma_F)\xi(1-k\xi)}(\xi-k\xi)+1-\frac{2B(\gamma_F)}{2A-B(\gamma_F)\xi(1-k\xi)})=$$
$$(1-c)k(\frac{2B(\gamma_F)(\xi-k\xi)+2A-B(\gamma_F)\xi(1-k\xi)-2B(\gamma_F)}{2A-B(\gamma_F)\xi(1-k\xi)})=$$
$$(1-c)k(\frac{(1+k\xi)(B(\gamma_F)\xi+2(A-B(\gamma_F)))}{2A-B(\gamma_F)\xi(1-k\xi)})\geq 0$$

(c) Next, we show that $p_F^*\geq p_L^*$.

$$p_F^*-p_L^*=\frac{(1-c)k\left(\beta_F^*\gamma_F\left(k\xi-\xi\beta_L^*\gamma_L+1\right)-\beta_L^*\gamma_L\right)}{\left(2k-\beta_L\gamma_L\right)\left(\beta_F^*\gamma_F(k\xi-1)+2k\right)} \tag{6}$$

The denominator of **??** is positive. Hence, we focus on the numerator. We need to show:
$\beta_F^*\gamma_F(k\xi+1)-\beta_L^*\gamma_L(\xi\beta_F^*\gamma_F+1)\geq 0$
We substitute $\beta_L^*\gamma_L=\frac{B(\gamma_L)}{A}$ and $\beta_F^*\gamma_F=\frac{2B(\gamma_F)}{2A-B(\gamma_F)\xi(1-k\xi)}$ to get:

$$\beta_F^*\gamma_F(k\xi+1)-\beta_L^*\gamma_L(\xi\beta_F^*\gamma_F+1)=$$
$$\frac{2B(\gamma_F)(k\xi+1)}{2A+B(\gamma_F)\xi(k\xi-1)}-\beta_L^*\gamma_L(\xi\frac{2B(\gamma_F)}{2A+B(\gamma_F)\xi(k\xi-1)}+1)=$$
$$\frac{2AB(\gamma_F)(k\xi+1)-B(\gamma_L)\left(\xi 2B(\gamma_F)+2A+B(\gamma_F)\xi(k\xi-1)\right)}{A(2A+B(\gamma_F)\xi(k\xi-1))}$$

Again, denominator is positive, and we focus on the numerator. Hence, we want to show: $2AB(\gamma_F)(k\xi+1)-B(\gamma_L)\left(\xi 2B(\gamma_F)+2A+B(\gamma_F)\xi(k\xi-1)\right)\geq 0$

$$2AB(\gamma_F)(k\xi+1)-B(\gamma_L)\left(\xi 2B(\gamma_F)+2A+B(\gamma_F)\xi(k\xi-1)\right)=$$
$$2AB(\gamma_F)(k\xi+1)-B(\gamma_L)B(\gamma_F)\xi(k\xi+1)-B(\gamma_L)2A=$$
$$B(\gamma_F)(k\xi+1)(2A-B(\gamma_L)\xi)-B(\gamma_L)2A>$$
[Next line follows since $B(\gamma_L)<B(\gamma_F)$:]
$$B(\gamma_F)(k\xi+1)(2A-B(\gamma_L)\xi)-B(\gamma_F)2A=$$
$$B(\gamma_F)\left((k\xi+1)(2A-B(\gamma_L)\xi)-2A\right)=$$
$$B(\gamma_F)\left(k\xi(2A-B(\gamma_L)\xi)-B(\gamma_L)\xi\right)=$$
$$B(\gamma_F)\left(2Ak\xi-B(\gamma_L)\xi(k\xi+1)\right)>$$

[Next line follows since $B(\gamma_L) < (1-\tau)$ and then $A = (k\xi+1)(1-\tau)$:]

$B(\gamma_F)\,(2Ak\xi - \xi(k\xi+1)(1-\tau)) = B(\gamma_F)\xi A(2k-1) > 0$

4. Next, we show that $\beta_L^* > \beta_F^*$.

$$\beta_L^* - \beta_F^* = \frac{2A\left(\gamma_F B\left(\gamma_L\right) - \gamma_L B\left(\gamma_F\right)\right) + \xi\gamma_F(k\xi-1)B\left(\gamma_F\right)B\left(\gamma_L\right)}{A\gamma_F\gamma_L\left(2A + \xi(k\xi-1)B\left(\gamma_F\right)\right)}$$

Since $k\xi > 1$ by Assumption 1 and $\gamma_F B\left(\gamma_L\right) - \gamma_L B\left(\gamma_F\right) = (1-t)\left(\gamma_F - \gamma_L\right) \geq 0$, $\beta_L^* - \beta_F^* > 0$.