Online Supplement for "Advertising in Asymmetric Competing Supply Chains"

Appendix A further studies properties for advertising effort levels, hybrid advertising structures, and all efforts. Appendix B includes all proofs for the main findings of the paper.

Appendix A

A.1 Properties for Advertising Effort Levels

This subsection explores additional properties for advertising effort levels. Corollary 1 addresses the case of manufacturer advertising and retailer advertising without cost sharing. Corollary 2 addresses the case of manufacturer advertising and retailer advertising with cost sharing. We consider how the advertising effort level responds to the change of channel substitutability θ and base demand ratio Ω.

Corollary 1 *Under advertising without cost sharing, we obtain the following properties.*

1. *Chain* 1*'s advertising effort level* e *increases with* Ω *(e.g.,* $\frac{\partial e^*_{MM-mi}}{\partial \Omega} > 0$ *and* $\frac{\partial e^*_{RR-ri}}{\partial \Omega} > 0$*), whereas Chain* 2*'s advertising effort level* e *decreases with* Ω *(e.g.,* $\frac{\partial e^*_{MM-mi}}{\partial \Omega} < 0$ *and* $\frac{\partial e^*_{RR-ri}}{\partial \Omega} < 0$*).*

2. *The advertising effort level* e *does not always increase with* θ*. More specifically,* $\frac{\partial e^*_{MM-mi}}{\partial \theta} > 0$ *iff* $\Omega > \Omega^{e-\theta}_{MM}$ *and* $\frac{\partial e^*_{RR-ri}}{\partial \theta} > 0$ *iff* $\Omega > \Omega^{e-\theta}_{RR}$*, where*

$$\Omega^{e-\theta}_{MM} = \frac{784 - 384\theta^2 - 2056\theta^4 + 2992\theta^6 - 1711\theta^8 + 460\theta^{10} - 48\theta^{12}}{4\theta\left(1092 - 3388\theta^2 + 4281\theta^4 - 2824\theta^6 + 1034\theta^8 - 200\theta^{10} + 16\theta^{12}\right)},$$
$$\Omega^{e-\theta}_{RR} = \frac{729 - 567\theta^2 - 2808\theta^4 + 5448\theta^6 - 4064\theta^8 + 1424\theta^{10} - 192\theta^{12}}{2\theta\left(2187 - 8640\theta^2 + 13812\theta^4 - 11472\theta^6 + 5280\theta^8 - 1280\theta^{10} + 128\theta^{12}\right)}.$$

Corollary 1 shows that a player's advertising effort increases with its own base demand but decreases with its rival's. A player's advertising effort increases with channel substitutability level (θ) if and only if the player has an advantage in market size; otherwise, increasing the advertising effort will intensify the competition level between the supply chains.

Corollary 2 *Under advertising with cost sharing given $\Omega = 1$, we obtain the following properties.*

1. *For CSMM, the advertising effort level increases with θ iff $\theta > \theta_{CSMM}$;*
2. *For CSRR, the advertising effort level increases with θ iff $\eta > \eta_{CSRR}$, where θ_{CSMM} and θ_{CSRR} are unique in the feasible domain, where*

$$\begin{aligned}\theta_{CSMM} &= \{\theta| -4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5=0\},\\ \eta_{CSRR} &= \frac{-6+31\theta+7\theta^2-28\theta^3-2\theta^4+8\theta^5+\sqrt{\theta^2+2\theta^3-8\theta^4}}{2\left(-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5\right)}.\end{aligned}$$

Corollary 2 shows that in CSMM, the advertising effort level increases if and only if the channel substitutability is sufficiently high, while in CSRR, the advertising effort level increases if and only if the cost sharing rate is very high.

A.2 Hybrid Advertising Structures

For completeness, we now turn our attention to *hybrid advertising structures*, in which the sole advertising provider in each supply chain need not be the same kind of firm as in the other supply chain. In other words, both the manufacturer and the retailer in each supply chain can freely decide whether or not to advertise. We label the two additional structures as follows: In MR, Manufacturer 1 advertises in supply chain 1 and Retailer 2 advertises in supply chain 2; In RM, Retailer 1 and Manufacturer 2 are the ones to advertise in their respective supply chains. The requisite profit functions follow from Eqs. (4) and (5) by setting $\mathbf{1}_{m1}=1$ and $\mathbf{1}_{r2}=1$ for MR, or $\mathbf{1}_{m2}=1$ and $\mathbf{1}_{r1}=1$ for RM, with all remaining indicators in each case set to zero.

Hybrid structures are more difficult to analyze than manufacturer/retailer advertising because of the interdependence of the decisions of the manufacturer and the retailer within each supply chain. For instance, with pure manufacturer advertising, Manufacturer 1 simply need only choose which of NM and MM provides itself with higher profit. However, in a hybrid structure, Manufacture 1 could consider abandoning advertising in anticipation that Retailer 1 would advertise. But this would require that Retailer 1 must profit more in RM than in MM; otherwise, the players would be at odds about which side should advertise. Said differently, Manufacturer 1 and Retailer 1 are a coalition in the sense that they have to coordinate on who advertises in order to obtain a mutual benefit. So we must compare the performance of different effort structures from a coalition's perspective.

To describe the stability of the advertising structure, we introduce the concept of *strong channel equilibrium*, in which no coalition of players within the same channel/supply chain can profitably deviate from the current state.[8] So, for MM to not be a strong channel equilibrium would mean that at least a manufacturer-retailer dyad would be better off by simultaneously defecting to either RM or MR.

Lemma 6 in the Appendix documents the comparison of MR and RM and the earlier advertising structures from the perspective of Manufacturer 1 and Retailer 1 as a coalition. Those findings lead to the following equilibrium results.

Theorem 5 *For hybrid advertising structures:*

1. *MM is a strong channel equilibrium if* $\hat{\Omega}_{r2}^{MR-MM}(\theta) < \Omega < \hat{\Omega}_{r1}^{RM-MM}(\theta)$ *in* $\theta \in [0.424, 0.823]$*;*
2. *RR is a strong channel equilibrium if* $\hat{\Omega}_{m1}^{MR-RR}(\theta) < \Omega < \hat{\Omega}_{m2}^{RM-RR}(\theta)$ *in* $\theta \in [0, 0.775]$*;*
3. *MR is a strong channel equilibrium if* $\Omega < \min\{\hat{\Omega}_{m2}^{MR-MM}(\theta), \max\{\hat{\Omega}_{r1}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\}\}$*;*
4. *RM is a strong channel equilibrium if* $\Omega > \max\{\hat{\Omega}_{m1}^{RM-MM}(\theta), \min\{\hat{\Omega}_{r2}^{RM-MM}(\theta), \hat{\Omega}_{m2}^{RM-MM}(\theta)\}\}$.

Figure 10 graphically illustrates Theorem 5.

[8]Strong channel equilibrium is a special case of strong equilibrium that limits the coalition to the players within the same supply chain. For a definition of strong equilibrium, please see Aumann (1959) and Bernheim et al. (1987).

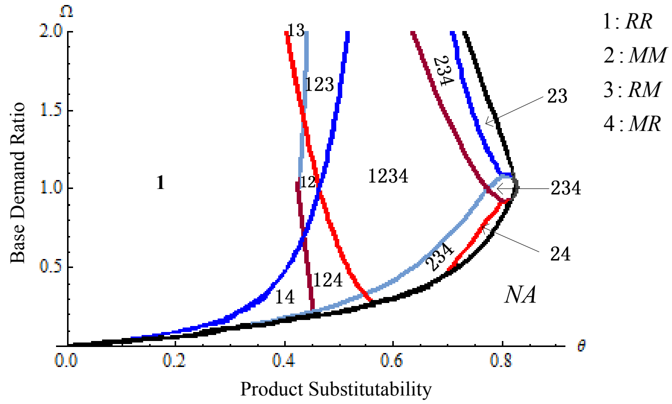

Figure 10: Equilibrium for hybrid advertising structures. 1 refers to RR, 2 to MM, 3 to RM, and 4 to MR. These numerical labels are a more compact way to present the equilibria for each region.

RR is the sole strong channel equilibrium if product substitutability is low because retailer advertising is more efficient in expanding the market, as well as reducing double marginalization. This is sufficient to offset any losses caused by intensified competition, when the supply chains are relatively monopolistic. As product substitutability grows, the advantages of RR erode but are sufficient to retain its equilibrium status unless product substitutability becomes too high (i.e., $\theta > 0.775$). MM exhibits stability as long as product substitutability is sufficiently high (i.e., $\theta > 0.424$). The strong channel equilibrium areas of MR and RM are asymmetric due to their advertising structure asymmetry. When a supply chain has the larger base demand, the supply chain is more likely to favor retailer advertising while the other supply chain sticks with the manufacturer's. Either MR or RM becomes unstable when product substitutability is sufficiently high and the supply chain with smaller base demand uses retailer advertising, because low retail prices and high effort costs force both players in the supply chain with smaller demand to switch to a more balanced advertising structure (i.e., MM). This confirms that manufacturer advertising is more stable when supply chain competition is intense, although the Prisoner's Dilemma persists.

A.3 All Efforts

The main body of this paper presents the analysis of advertising that is performed solely by either the manufacturer or the retailer in each supply chain. We now consider the scenario in which manufacturers and retailers advertise simultaneously, which we call *all efforts* (AE).

Let e_{mi} denote the advertising by Manufacturer $i, i = 1, 2$, and e_{ri} denote the advertising by Retailer $i, i = 1, 2$. We adapt Eq. (1)'s representation of base demand in channel i to become

$$\alpha_i = A_i + e_{mi} + e_{ri}.$$

This additive form, used for reasons of tractability, does not capture any diminishing returns when manufacturers and retailers both advertise to the same target market (Venkatesh and Kamakura, 2003), or any synergies for that matter.

The players' profit functions are given by, for i=1,2,

$$\begin{aligned} \Pi_{mi} &= D_i w_i - k_{mi} e_{mi}^2, \\ \Pi_{ri} &= D_i (p_i - w_i) - k_{ri} e_{ri}^2, \end{aligned}$$

where k_{mi} and k_{ri} are cost coefficients for the efforts of manufacturers and retailers, respectively. In the game, the manufacturers simultaneously determine wholesale prices w_i and effort levels e_{mi} in the first stage, and in the second the retailers simultaneously determine retail prices p_i and advertising levels e_{ri}. To simplify the following analysis we set $A_1 = A_2 = 1$ and $k_{m2} = k_{r1} = k_{r2} = 1$, while focusing on changes in the value of k_{m1}.

Lemma 5 *Given $A_1 = A_2 = 1$ and $k_{12} = k_{21} = k_{22} = 1$, Manufacturer 1's advertising effort decreases with its cost coefficient (k_{m1}).*

The proof of Lemma 5 also indicates that as the cost coefficient goes to infinity, the corresponding advertising level converges to zero. In our numerical analysis this property emerges for all players without the restrictions $A_1 = A_2 = 1$ and $k_{12} = k_{21} = k_{22} = 1$. Further, all else equal, the player with the lower cost coefficient will exert the higher advertising effort.

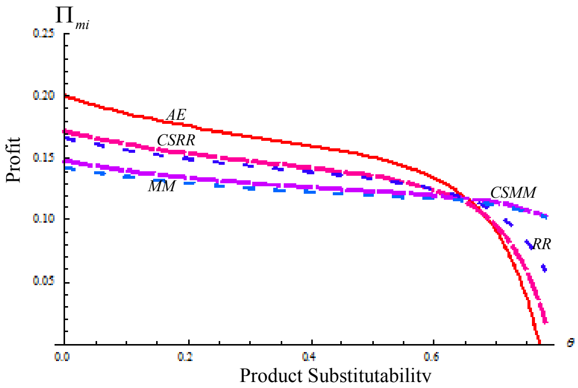

Figure 11: Manufacturers' profit comparison among AE, CSRR, CSMM, MM, and RR, given $\Omega = 1$ and $\eta = 0.2$

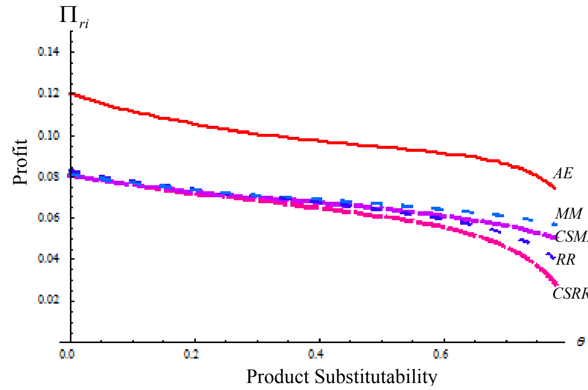

Figure 12: Retailers' profit comparison among AE, CSRR, CSMM, MM, and RR, given $\Omega = 1$ and $\eta = 0.2$

We now numerically compare AE to the previously analyzed advertising games. Note that the following representative example will convey the major qualitative insights even if the parameter values are changed. When channel substitutability is relatively low, AE outperforms MM, RR, CSMM, and CSRR for the manufacturers (Figure 11) whereas it dominates all other structures for the retailers (Figure 12). We also find that AE could be more preferable to retailers than manufacturers, because AE imposes more effort costs upon the manufacturers than upon the retailers. AE could perform worse than other advertising structures for the manufacturers if channel substitutability is substantially high. This is because AE results in more combined efforts than any other game, which significantly intensifies horizontal channel competition and incites a pricing war between the channels.

We include the AE analysis for the sake of completeness, although its complexity limits the availability of generalizable insights. In any case, this paper's main model is better suited to address our central research questions, whose industry motivations are presented in detail in Section 1. By focusing on manufacturer-only or retailer-only advertising, while allowing cost sharing, we can more sharply illuminate the impact of where control of the advertising decision is located in the supply chain, and the interplay between that control and the source of the advertising's funding in a competitive setting.

Appendix B

In our notation the index i $(i = 1, 2)$ identifies the channel or supply chain. Unless indicated otherwise, all equations below hold for $i = 1, 2$.

Proof of Lemma 1: To compare MM, MN, NM, and NN, we solve each subgame by reverse induction. More specifically, we first compute the retailers' best-response retail prices, then substitute them into the manufacturers' profit functions, and finally solve the manufacturers' first-order conditions for wholesale prices and advertising levels. Each subgame has a unique equilibrium. Comparing the manufacturers' profits across all subgames yields the subgame perfect equilibrium for the whole game.

In MM, given w_i and e_i, Retailer i's profits are concave with respect to p_i because $\frac{\partial^2 \Pi_{MM-ri}}{\partial p_i^2} = -\frac{2}{1-\theta^2} < 0$˙The best response retail price function can be obtained by solving from the first-order condition.

$$p_i(w_i, e_i) = \frac{(2-\theta^2)(A_i + e_i) - \theta(A_{3-i} + e_{3-i}) + 2w_i + \theta w_{3-i}}{4-\theta^2}, i = 1, 2.$$

Then, substituting $p_i(w_i, e_i)$ into the manufacturers' profit functions, we get

$$\Pi_{MM-mi}(w_i, e_i) = \frac{\left(2-\theta^2\right) q_i w_i + w_i \left(\left(2-\theta^2\right)\left(A_i - w_i\right) - \theta\left(A_{3-i} + q_{3-i} - w_{3-i}\right)\right) - (1-\theta^2)(4-\theta^2) q_i^2}{(1-\theta)(4-\theta)}.$$

The corresponding Hessian matrix is negative definite because

$$\frac{\partial^2 \Pi_{MM-mi}(w_i, e_i)}{\partial w_i^2} = -\frac{2\left(2-\theta^2\right)}{\left(1-\theta^2\right)\left(4-\theta^2\right)} < 0$$

and

$$\begin{aligned} & \frac{\partial^2 \Pi_{MM-mi}(w_i, e_i)}{\partial w_i^2} \frac{\partial^2 \Pi_{MM-mi}(w_i, e_i)}{\partial e_i^2} - \frac{\partial^2 \Pi_{MM-mi}(w_i, e_i)}{\partial w_i \partial e_i} - \frac{\partial^2 \Pi_{MM-mi}(w_i, e_i)}{\partial e_i \partial w_i} \\ = \quad & \frac{28 - 52\theta^2 + 27\theta^4 - 4\theta^6}{(4 - 5\theta^2 + \theta^4)^2} > 0. \end{aligned}$$

So, we can obtain the optimal w^*_{MM-i} and e^*_{MM-mi}. Replacing them into $p_i(w_i, e_i)$produces the optimal retail prices p^*_{MM-i}.

In summary, the unique equilibrium for MM is:

$$w^*_{MM-i} \quad = \quad \frac{2\left(4 - 5\theta^2 + \theta^4\right)\left(\left(14 - 17\theta^2 + 4\theta^4\right) A_i - 2\theta\left(2-\theta^2\right) A_{3-i}\right)}{196 - 492\theta^2 + 417\theta^4 - 140\theta^6 + 16\theta^8},$$

$$
\begin{aligned}
p^*_{MM-i} &= \frac{4\left(3-4\theta^2+\theta^4\right)\left(\left(14-17\theta^2+4\theta^4\right)A_i-2\theta\left(2-\theta^2\right)A_{3-i}\right)}{196-492\theta^2+417\theta^4-140\theta^6+16\theta^8},\\
e^*_{MM-mi} &= \frac{\left(2-\theta^2\right)\left(\left(14-17\theta^2+4\theta^4\right)A_i-2\theta\left(2-\theta^2\right)A_{3-i}\right)}{196-492\theta^2+417\theta^4-140\theta^6+16\theta^8},\\
D^*_{MM-i} &= \frac{2\left(2-\theta^2\right)\left(\left(14-17\theta^2+4\theta^4\right)A_i-2\theta\left(2-\theta^2\right)A_{3-i}\right)}{196-492\theta^2+417\theta^4-140\theta^6+16\theta^8},\\
\Pi^*_{MM-mi} &= \frac{(2-\theta^2)(14-19\theta^2+4\theta^4)((14-17\theta^2+4\theta^4)A_i-2\theta(2-\theta^2)A_{3-i})^2}{(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8)^2},\\
\Pi^*_{MM-ri} &= \frac{4(2-\theta^2)^2(1-\theta^2)((14-17\theta^2+4\theta^4)A_i-2\theta(2-\theta^2)A_{3-i})^2}{(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8)^2}.
\end{aligned}
$$

For prices and demands to remain nonnegative requires

$$\left(14-17\theta^2+4\theta^4\right)A_i-2\theta\left(2-\theta^2\right)A_{3-i}\geq 0.$$

This is equivalent to $\frac{2\theta\left(2-\theta^2\right)}{14-17\theta^2+4\theta^4}\leq\Omega\leq\frac{14-17\theta^2+4\theta^4}{2\theta(2-\theta^2)}$, where the maximum feasible domain for θ is given by $[0,0.940]$ because the upper bound of θ is obtained when the above two constraint boundary lines cross at $A_1=A_2$.

For subgame NN, given the wholesale prices w_i, Retailer i's profit is concave with respect to p_i because $\frac{\partial^2\Pi_{NN-ri}}{\partial p_i^2}=-\frac{2}{1-\theta^2}<0$. The response function of the retail prices can be obtained by solving the following first-order conditions.

$$p_i(w_i)=\frac{\left(2-\theta^2\right)A_i-\theta A_{3-i}+2w_i+\theta w_{3-i}}{4-\theta^2}, i=1,2.$$

Substituting $p_i(w_i)$ into the manufacturers' profit function yields

$$\Pi_{NN-mi}(w_i)=\frac{w_i\left(\left(2-\theta^2\right)A_i-\theta A_{3-i}-2w_i+\theta^2w_i+\theta w_{3-i}\right)}{4-5\theta^2+\theta^4}.$$

Manufacturer i's profit, $\Pi_{NN-mi}(w_i)$, is concave in w_i because $\frac{\partial^2\Pi_{NN-mi}}{\partial w_i^2}=-\frac{2\left(2-\theta^2\right)}{4-5\theta^2+\theta^4}<0$. So, we can obtain the unique and optimal wholesale prices w^*_{NN-i}. Substituting these into $p_i(w_i)$ delivers p^*_{NN-i}.

The unique subgame perfect equilibrium for NN is:

$$
\begin{aligned}
w^*_{NN-i} &= \frac{\left(8-9\theta^2+2\theta^4\right)A_i-\theta\left(2-\theta^2\right)A_{3-i}}{16-17\theta^2+4\theta^4},\\
p^*_{NN-i} &= \frac{2\left(3-\theta^2\right)\left(\left(8-9\theta^2+2\theta^4\right)A_i-\theta\left(2-\theta^2\right)A_{3-i}\right)}{64-84\theta^2+33\theta^4-4\theta^6},
\end{aligned}
$$

$$
\begin{aligned}
D^*_{NN-i} &= \frac{(2-\theta^2)\left((8-9\theta^2+2\theta^4)A_i-\theta(2-\theta^2)A_{3-i}\right)}{64-148\theta^2+117\theta^4-37\theta^6+4\theta^8}, \\
\Pi^*_{NN-mi} &= \frac{(2-\theta^2)\left((8-9\theta^2+2\theta^4)A_i-\theta(2-\theta^2)A_{3-i}\right)^2}{(4-5\theta^2+\theta^4)\left(16-17\theta^2+4\theta^4\right)^2}, \\
\Pi^*_{NN-ri} &= \frac{(2-\theta^2)^2\left((8-9\theta^2+2\theta^4)A_i-\theta(2-\theta^2)A_{3-i}\right)^2}{(1-\theta^2)\left(64-84\theta^2+33\theta^4-4\theta^6\right)^2}.
\end{aligned}
$$

For prices and demands to remain nonnegative requires

$$
\left(8-9\theta^2+2\theta^4\right)A_i-\theta\left(2-\theta^2\right)A_{3-i}\geq 0.
$$

This is equivalent to $\frac{\theta\left(2-\theta^2\right)}{8-9\theta^2+2\theta^4}\leq\Omega\leq\frac{8-9\theta^2+2\theta^4}{\theta(2-\theta^2)}$. The maximum feasible domain for θ is given by $\theta\in[0,1]$ as the upper bound of θ is obtained when the two constraint boundary lines cross.

For subgame MN, given w_i and e_1, Retailer i's profits are concave in p_i, because $\frac{\partial^2\Pi_{MN-ri}}{\partial p_i^2}=-\frac{1}{1-\theta^2}<0$. The best response retail prices derived from the first order conditions are

$$
\begin{aligned}
p_1(w_i,e_1) &= \frac{(2-\theta^2)A_1-\theta A_2+2e_1-\theta^2e_1+2w_1+\theta w_2}{4-\theta^2}; \\
p_2(w_i,e_1) &= \frac{(2-\theta^2)A_2-\theta A_1-\theta w_1+\theta w_1+2w_2}{4-\theta^2}.
\end{aligned}
$$

Substituting $p_i(w_i,e_1)$ into the manufacturers' profit functions yields

$$
\begin{aligned}
\Pi_{MN-m1}(w_i,e_1) &= \frac{-(4-5\theta^2+\theta^4)e_1^2+(2-\theta^2)e_1w_1+w_1\left((2-\theta^2)A_1-\theta A_2-2w_1+\theta^2w_1+\theta w_2\right)}{4-5\theta^2+\theta^4}; \\
\Pi_{MN-m2}(w_i,e_1) &= \frac{w_2\left(-\theta A_1+(2-\theta^2)A_2-\theta e_1+\theta w_1-2w_2+\theta^2w_2\right)}{4-5\theta^2+\theta^4}.
\end{aligned}
$$

The $\Pi_{MN-m1}(w_i,e_1)$ are concave on (w_1,e_1) because $\frac{\partial^2\Pi_{MN-m1}(w_i,e_1)}{\partial w_1^2}=-\frac{2(2-\theta^2)}{4-5\theta^2+\theta^4}<0$ and the second-order Hessian Matrix has determinant $\frac{\partial^2\Pi_{MN-m1}(w_i,e_1)}{\partial w_1^2}\frac{\partial^2\Pi_{MN-m1}(w_i,e_1)}{\partial e_1^2}-\frac{\partial^2\Pi_{MN-m1}(w_i,e_1)}{\partial w_1\partial e_1}\frac{\partial^2\Pi_{MN-m1}(w_i,e_1)}{\partial e_1\partial w_1}=\frac{28-52\theta^2+27\theta^4-4\theta^6}{(4-5\theta^2+\theta^4)^2}$, which is strictly positive in the feasible domain of $\theta\in[0,0.94]$. Meanwhile, $\Pi_{MN-m2}(w_i,e_1)$ is concave on w_2 because $\frac{\partial^2\Pi_{MN-m2}(w_2)}{\partial w_1^2}=-\frac{2(2-\theta^2)}{4-5\theta^2+\theta^4}<0$. So, we can obtain the unique equilibrium wholesale prices and advertising level

$$
\begin{aligned}
w^*_{MN-1} &= \frac{2(4-5\theta^2+\theta^4)\left((8-9\theta^2+2\theta^4)A_1+\theta(-2+\theta^2)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8}; \\
w^*_{MN-2} &= \frac{(4-5\theta^2+\theta^4)\left(2\theta(-2+\theta^2)A_1+(14-17\theta^2+4\theta^4)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8}; \\
e^*_{MN-1} &= \frac{(2-\theta^2)\left((8-9\theta^2+2\theta^4)A_1-\theta(2-\theta^2)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8},
\end{aligned}
$$

and substituting these into $p_i(w_i, e_1)$ yields the following equilibrium retail prices

$$
\begin{aligned}
p^*_{MN-1} &= \frac{4\left(3-4\theta^2+\theta^4\right)\left(\left(8-9\theta^2+2\theta^4\right)A_1+\theta\left(-2+\theta^2\right)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8};\\
p^*_{MN-2} &= \frac{2\left(3-4\theta^2+\theta^4\right)\left(2\theta\left(-2+\theta^2\right)A_1+\left(14-17\theta^2+4\theta^4\right)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8}.
\end{aligned}
$$

A similar process obtains the following demands and profits for Manufacturer 1 in MN and NM, [9]

$$
\begin{aligned}
D^*_{MN-1} &= \frac{2\left(2-\theta^2\right)\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8},\\
D^*_{MN-2} &= \frac{\left(2-\theta^2\right)\left(\left(14-17\theta^2+4\theta^4\right)A_2-2\theta\left(2-\theta^2\right)A_1\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8},\\
\Pi^*_{MN-m1} &= \frac{\left(2-\theta^2\right)\left(14-19\theta^2+4\theta^4\right)\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2},\\
D^*_{NM-1} &= \frac{\left(2-\theta^2\right)\left(\left(14-17\theta^2+4\theta^4\right)A_1-2\theta\left(2-\theta^2\right)A_2\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8},\\
D^*_{NM-2} &= \frac{2\left(2-\theta^2\right)\left(\left(8-9\theta^2+2\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)}{112-270\theta^2+221\theta^4-72\theta^6+8\theta^8},\\
\Pi^*_{NM-m1} &= \frac{\left(4-\theta^2\right)\left(2-3\theta^2+\theta^4\right)\left(\left(14-17\theta^2+4\theta^4\right)A_1-2\theta\left(2-\theta^2\right)A_2\right)^2}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}.
\end{aligned}
$$

In the following, without loss of generality, we compare Manufacturer 1's profits across the various cases. To compare MN and NN, we use $\Delta\Pi^{MN-NN}_{m1}$ to denote Manufacturer 1's profit in MN minus its profit in NN. The earlier profit expressions yield

$$
\Delta\Pi^{MN-NN}_{m1} = \frac{\left(2-\theta^2\right)^2\left(896-3232\theta^2+4570\theta^4-3222\theta^6+1191\theta^8-220\theta^{10}+16\theta^{12}\right)\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(4-5\theta^2+\theta^4\right)\left(16-17\theta^2+4\theta^4\right)^2\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}.
$$

The common lower and upper bounds of the constrained areas are defined by

$$
\underline{\Omega}^{MN-NN}(\theta) = \frac{\theta\left(2-\theta^2\right)}{8-9\theta^2+2\theta^4} \quad \text{and} \quad \bar{\Omega}^{MN-NN}(\theta) = \frac{\left(14-17\theta^2+4\theta^4\right)}{2\theta\left(2-\theta^2\right)},
$$

where the domain for θ is $\theta \in [0, 0.967]$. Then $\Delta\Pi^{MN-NN}_{m1} > 0$ as long as $896-3232\theta^2+4570\theta^4-3222\theta^6+1191\theta^8-220\theta^{10}+16\theta^{12} > 0$, which is always true in its feasible domain.

A similar approach shows for the comparison of NM with NN that

$$
\begin{aligned}
\Delta\Pi^{NM-NN}_{m1} = &-\frac{\left(2-\theta^2\right)\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(4-5\theta^2+\theta^4\right)\left(16-17\theta^2+4\theta^4\right)^2}\\
&-\frac{\left(4-\theta^2\right)\left(2-3\theta^2+\theta^4\right)\left(\left(14-17\theta^2+4\theta^4\right)A_1-2\theta\left(2-\theta^2\right)A_2\right)^2}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}
\end{aligned}
$$

[9] The values for Manufacturer 2 can be obtained by replacing every 1 with 2 and vice versa. Other results are omitted for brevity.

$$< \quad 0,$$

This is supported by the common lower and upper bounds

$$\underline{\Omega}^{NM-NN}(\theta) = \frac{2\theta\left(2-\theta^2\right)}{14-17\theta^2+4\theta^4} \quad \text{and} \quad \bar{\Omega}^{NM-NN}(\theta) = \frac{\left(8-9\theta^2+2\theta^4\right)}{\theta\left(2-\theta^2\right)},$$

where $\theta \in [0, 0.967]$. As before, the upper limit for θ is obtained when the two constraint lines, $\underline{\Omega}^{NM-NN}(\theta)$ and $\bar{\Omega}^{NM-NN}(\theta)$, cross.

For the comparison of MM with NM we have

$$\begin{aligned}\Delta\Pi_{m1}^{MM-NM} &= \left(\frac{28-52\theta^2+27\theta^4-4\theta^6}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2} - \frac{\left(4-\theta^2\right)\left(2-3\theta^2+\theta^4\right)}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}\right) \\ &\quad \times \left(\left(14-17\theta^2+4\theta^4\right)A_1 - 2\theta\left(2-\theta^2\right)A_2\right)^2.\end{aligned}$$

The expression is strictly positive since $\frac{28-52\theta^2+27\theta^4-4\theta^6}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2} > \frac{\left(4-\theta^2\right)\left(2-3\theta^2+\theta^4\right)}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}$ for any $\theta \in [0, 0.940]$ as required by MM.

For MM and MN we have

$$\begin{aligned}&\Delta\Pi_{m1}^{MM-MN} \\ &= \left(2-\theta^2\right)\left(14-19\theta^2+4\theta^4\right)\left(\frac{\left(\left(14-17\theta^2+4\theta^4\right)A_1-2\theta\left(2-\theta^2\right)A_2\right)^2}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2} - \frac{\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(112-270\theta^2+221\theta^4-72\theta^6+8\theta^8\right)^2}\right).\end{aligned}$$

This is strictly negative between $\frac{2\theta\left(2-\theta^2\right)}{14-17\theta^2+4\theta^4}$ and $\frac{\left(14-17\theta^2+4\theta^4\right)}{2\theta\left(2-\theta^2\right)}$.

This progression indicates that Manufacturer 1 always benefits from providing advertising effort regardless of what the other manufacturer does, but is harmed by the other manufacturer's choice to advertise. The same techniques provide the corresponding results for Manufacturer 2. □

Proof of Theorem 1: The first part of Theorem 1 results directly from Lemma 1. The Prisoner's Dilemma can be demonstrated by comparing Manufacturer 1's profits in MM and NN. It is easy to show that the common feasible area of MM and NN is confined by MM's feasible area. Therefore,

$$\underline{\Omega}^{MM-NN}(\theta) = \frac{2\theta\left(2-\theta^2\right)}{14-17\theta^2+4\theta^4} \quad \text{and} \quad \bar{\Omega}^{MM-NN}(\theta) = \frac{\left(14-17\theta^2+4\theta^4\right)}{2\theta\left(2-\theta^2\right)}.$$

A special case is given by $\theta \in [0, 0.940]$ when the above two constraint lines cross. [10]

[10]The feasible range for θ becomes smaller as the base demand ratio diverges, as illustrated in Figure 1.

Define $\Delta\Pi_{m1}^{MM-NN}$ as Manufacturer 1's profit in MM minus its profit in NN. We have

$$\begin{aligned}\Delta\Pi_{m1}^{MM-NN} &= \left(2-\theta^2\right)\Bigg(\frac{\left(14-19\theta^2+4\theta^4\right)\left(\left(14-17\theta^2+4\theta^4\right)A_1-2\theta\left(2-\theta^2\right)A_2\right)^2}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2} \\ &\quad -\frac{\left(\left(8-9\theta^2+2\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(4-5\theta^2+\theta^4\right)\left(16-17\theta^2+4\theta^4\right)^2}\Bigg) \qquad \text{(A-1)}\end{aligned}$$

Making the change of variable $\Omega = A_1/A_2$ and solving $\Delta\Pi_{m1}^{MM-NN} = 0$ yields two roots:

$$\begin{aligned}\hat{\Omega}_{m1-1}^{MM-NN}(\theta) &= \frac{K_1 + K_2\sqrt{56-146\theta^2+125\theta^4-39\theta^6+4\theta^8}}{175616-1034880\theta^2+K_3}, \\ \hat{\Omega}_{m1-2}^{MM-NN}(\theta) &= \frac{K_1 - K_2\sqrt{56-146\theta^2+125\theta^4-39\theta^6+4\theta^8}}{175616-1034880\theta^2+K_3},\end{aligned}$$

where

$$\begin{aligned}K_1 &= 940800\theta-4982888\theta^3+11381144\theta^5-14694456\theta^7+11805776\theta^9-6117970\theta^{11}+2045720\theta^{13}-426080\theta^{15}+50240\theta^{17}-2560\theta^{19}, \\ K_2 &= \theta\left(6272-25544\theta^2+42844\theta^4-38414\theta^6+19905\theta^8-5968\theta^{10}+960\theta^{12}-64\theta^{14}\right), \\ K_3 &= 2677288\theta^4-3997072\theta^6+3806878\theta^8-2413562\theta^{10}+1031035\theta^{12}-293184\theta^{14}+53184\theta^{16}-5568\theta^{18}+256\theta^{20}.\end{aligned}$$

Since $\hat{\Omega}_{m1-2}^{MM-NN}(\theta)$ is below the common lower bound in cases MM and NN, we define

$$\hat{\Omega}_{m1}^{MM-NN}(\theta) = \min\{\hat{\Omega}_{m1-1}^{MM-NN}(\theta), \bar{\Omega}^{MM-NN}(\theta)\},$$

which is the boundary line for Manufacturer 1's preferences between MM and NN (shown in Figure 1). Note that $\hat{\Omega}_{m1-1}^{MM-NN}(\theta) \le 1$ in $\theta \in [0, 0.940]$.

Similarly, we can define

$$\hat{\Omega}_{m2}^{MM-NN}(\theta) = \min\{\hat{\Omega}_{m2-1}^{MM-NN}(\theta), \underline{\Omega}^{MM-NN}(\theta)\}$$

for Manufacturer 2 (shown in Figure 1), where

$$\hat{\Omega}_{m2-1}^{MM-NN}(\theta) = \frac{K_1 + K_2\sqrt{56-146\theta^2+125\theta^4-39\theta^6+4\theta^8}}{\theta^2(\theta^2-2)^2 K_4},$$

and

$$K_4 = 9464 - 34516\theta^2 + 49530\theta^4 - 35595\theta^6 + 13476\theta^8 - 2560\theta^{10} + 192\theta^{12}.$$

Here we also characterize the monotonicity of optimal retail prices and demand with respect to θ within the common feasible range of θ in the different subgames. We consider only subgames NN and MM, as the others are similar. In NN,

$$\frac{\partial p_{NN-i}^*}{\partial\theta} = -\frac{4\theta\left(224-336\theta^2+201\theta^4-56\theta^6+6\theta^8\right)A_i}{\left(64-84\theta^2+33\theta^4-4\theta^6\right)^2}$$

$$- \quad \frac{2\left(384-456\theta^2+146\theta^4+33\theta^6-27\theta^8+4\theta^{10}\right)A_{3-i}}{\left(64-84\theta^2+33\theta^4-4\theta^6\right)^2}.$$

This is strictly negative because $224-336\theta^2+201\theta^4-56\theta^6+6\theta^8>0$ and $384-456\theta^2+146\theta^4+33\theta^6-27\theta^8+4\theta^{10}>0$ for any $\theta\in[0,1)$. Also

$$\begin{aligned}\frac{\partial D^*_{NN-i}}{\partial\theta} &= \frac{2\theta\left(704-2080\theta^2+2510\theta^4-1588\theta^6+559\theta^8-104\theta^{10}+8\theta^{12}\right)A_i}{\left(64-148\theta^2+117\theta^4-37\theta^6+4\theta^8\right)^2}\\ &- \frac{\left(256-176\theta^2-492\theta^4+764\theta^6-439\theta^8+117\theta^{10}-12\theta^{12}\right)A_{3-i}}{\left(64-148\theta^2+117\theta^4-37\theta^6+4\theta^8\right)^2}.\end{aligned}$$

This indicates that D^*_{NN-i} increases with θ if $\Omega>\frac{256-176\theta^2-492\theta^4+764\theta^6-439\theta^8+117\theta^{10}-12\theta^{12}}{2\theta(704-2080\theta^2+2510\theta^4-1588\theta^6+559\theta^8-104\theta^{10}+8\theta^{12})}$ but decreases with θ otherwise. So, if the supply chains are sufficiently asymmetric, the supply chain with the larger base market obtains more demand as product substitutability grows.

For MM,

$$\begin{aligned}\frac{\partial p^*_{MM-i}}{\partial\theta} = -&\quad \frac{8\theta\left(308-1820\theta^2+3393\theta^4-2960\theta^6+1369\theta^8-328\theta^{10}+32\theta^{12}\right)A_i}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2}\\ &- \frac{8\left(1176-3516\theta^2+3786\theta^4-1441\theta^6-330\theta^8+469\theta^{10}-148\theta^{12}+16\theta^{14}\right)A_{3-i}}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2}.\end{aligned}$$

This is strictly negative for any θ in the feasible range, ensuring nonnegative prices and demands.

$$\begin{aligned}\frac{\partial D^*_{MM-i}}{\partial\theta} &= \frac{16\theta\left(1092-3388\theta^2+4281\theta^4-2824\theta^6+1034\theta^8-200\theta^{10}+16\theta^{12}\right)A_i}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2}\\ &- \frac{4\left(784-384\theta^2-2056\theta^4+2992\theta^6-1711\theta^8+460\theta^{10}-48\theta^{12}\right)A_{3-i}}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2}.\end{aligned}$$

D^*_{MM-i} increases with θ if $\Omega>\frac{784-384\theta^2-2056\theta^4+2992\theta^6-1711\theta^8+460\theta^{10}-48\theta^{12}}{4\theta(1092-3388\theta^2+4281\theta^4-2824\theta^6+1034\theta^8-200\theta^{10}+16\theta^{12})}$, but decreases with θ otherwise. We can show comparable properties for the other subgames in a similar fashion. □

Proof of Lemma 2: This Lemma's proof is similar to that of Lemma 1.

More specifically, we first compute the retailers' best-response retail prices and advertising levels, then substitute them into the manufacturers' profit functions, and finally solve the manufacturers' first-order condition for wholesale prices. Each subgame has a unique equilibrium. Comparing the retailers' profits across all subgames gives the subgame perfect equilibrium for the entire game.

Here we start with RR. Other subgames can be solved similarly. Given w_i, the retailers' profits are jointly concave in p_i and e_i because $\frac{\partial\Pi^2_{RR-ri}(w_i)}{\partial p_i^2}=-\frac{2}{1-\theta^2}<0$ and the determinant of its

Hessian matrix

$$
\begin{aligned}
& \frac{\partial^2\Pi_{RR-ri}(w_i)}{\partial p_i^2}\frac{\partial^2\Pi_{RR-ri}(w_i)}{\partial e_i^2} - \frac{\partial^2\Pi_{RR-ri}(w_i)}{\partial p_i \partial e_i}\frac{\partial^2\Pi_{RR-ri}(w_i)}{\partial e_i \partial p_i} \\
= & \frac{3-4\theta^2}{(1-\theta^2)^2} \\
> & \; 0
\end{aligned}
$$

as long as $\theta < \frac{\sqrt{3}}{2}$, which is true in the feasible domain.

According to the first-order conditions,

$$
\begin{aligned}
p_i(w_i) &= \frac{2\left(3-5\theta^2+2\theta^4\right)A_i + 4\theta\left(-1+\theta^2\right)A_{3-i} + 3w_i - 6\theta^2 w_i + 4\theta w_{3-i} - 4\theta^3 w_{3-i}}{9-16\theta^2+4\theta^4}; \\
e_i(w_i) &= \frac{\left(3-2\theta^2\right)A_i - 2\theta A_{3-i} - 3w_i + 2\theta^2 w_i + 2\theta w_{3-i}}{9-16\theta^2+4\theta^4}.
\end{aligned}
$$

Substituting $p_i(w_i)$ and $e_i(w_i)$ into the manufacturers' profit functions yields

$$
\Pi_{RR-mi}(w_i) = \frac{2w_i\left(\left(3-2\theta^2\right)A_i - 2\theta A_{3-i} - 3w_i + 2\theta^2 w_i + 2\theta w_{3-i}\right)}{9-16\theta^2+4\theta^4}.
$$

$\Pi_{RR-mi}(w_i)$ is concave in w_i because $\frac{\partial^2\Pi_{RR-mi}(w_i)}{\partial w_i^2} = -\frac{4\left(3-2\theta^2\right)}{9-16\theta^2+4\theta^4} < 0$ as long as $\theta < \frac{\sqrt{9-\sqrt{33}}}{2}$, which is true in the feasible domain. Therefore, the equilibrium wholesale prices w^*_{RR-i} are unique. The equilibrium retail prices p^*_{RR-i} and advertising levels e^*_{RR-ri} also follow from the equilibrium wholesale prices.

In summary, the unique equilibrium for RR is:

$$
\begin{aligned}
w^*_{RR-i} &= \frac{\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}}{18-26\theta^2+8\theta^4}, \\
p^*_{RR-i} &= \frac{\left(15-26\theta^2+8\theta^4\right)\left(\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}\right)}{162-522\theta^2+560\theta^4-232\theta^6+32\theta^8}, \\
e^*_{RR-ri} &= \frac{\left(3-2\theta^2\right)\left(\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}\right)}{162-522\theta^2+560\theta^4-232\theta^6+32\theta^8}, \\
D^*_{RR-i} &= \frac{\left(3-2\theta^2\right)\left(\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}\right)}{81-261\theta^2+280\theta^4-116\theta^6+16\theta^8}, \\
\Pi^*_{RR-ri} &= \frac{\left(3-2\theta^2\right)^2\left(3-4\theta^2\right)\left(\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}\right)^2}{4\left(81-261\theta^2+280\theta^4-116\theta^6+16\theta^8\right)^2}, \\
\Pi^*_{RR-mi} &= \frac{\left(3-2\theta^2\right)\left(\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i}\right)^2}{2\left(9-16\theta^2+4\theta^4\right)\left(9-13\theta^2+4\theta^4\right)^2}.
\end{aligned}
$$

For the prices and demands in RR to be nonnegative requires

$$
\left(9-14\theta^2+4\theta^4\right)A_i - \theta\left(3-2\theta^2\right)A_{3-i} \geq 0.
$$

This is equivalent to $\frac{\theta\left(3-2\theta^2\right)}{9-14\theta^2+4\theta^4} \leq \Omega \leq \frac{9-14\theta^2+4\theta^4}{\theta(3-2\theta^2)}$, which implies the largest feasible domain for θ is given by $\theta \in [0, 0.823]$ and the upper bound of θ is reached when the above two constraint boundaries cross. Define

$$\underline{\Omega}^{RR} \equiv \frac{\theta\left(3-2\theta^2\right)}{9-14\theta^2+4\theta^4} \quad \text{and} \quad \bar{\Omega}^{RR} \equiv \frac{9-14\theta^2+4\theta^4}{\theta\left(3-2\theta^2\right)}.$$

The above constraint is the strictest of all cases in this paper.

In RN, given w_i, Retailer 1's profit is concave on (p_1, e_1) because $\frac{\partial^2 \Pi_{RN-r1}}{\partial p_1^2} = -\frac{2}{1-\theta^2} < 0$ and the second Hessian Matrix has determinant $\frac{\partial^2 \Pi_{RN-r1}}{\partial p_1^2}\frac{\partial^2 \Pi_{RN-r1}}{\partial e_1^2} - \frac{\partial^2 \Pi_{RN-r1}}{\partial p_1 \partial e_1}\frac{\partial^2 \Pi_{RN-r1}}{\partial e_1 \partial p_1} = \frac{3-4\theta^2}{(1-\theta^2)^2} > 0$, as long as $\theta < \frac{\sqrt{3}}{2}$ which is true on the common domain. Retailer 2's profit is concave on p_2 because $\frac{\partial^2 \Pi_{RN-r2}}{\partial p_2^2} = -\frac{2}{1-\theta^2} < 0$. The first-order conditions then yield

$$\begin{aligned}
p_1(w_1,w_2) &= \frac{2\left(2-3\theta^2+\theta^4\right)A_1 + 2\theta\left(-1+\theta^2\right)A_2 + 2w_1 - 3\theta^2 w_1 + 2\theta w_2 - 2\theta^3 w_2}{6-9\theta^2+2\theta^4}; \\
p_2(w_1,w_2) &= \frac{2\theta\left(-1+\theta^2\right)A_1 + \left(3-5\theta^2+2\theta^4\right)A_2 + 2\theta w_1 - 2\theta^3 w_1 + 3w_2 - 4\theta^2 w_2}{6-9\theta^2+2\theta^4}; \\
e_1(w_1,w_2) &= \frac{\left(2-\theta^2\right)A_1 - \theta A_2 - 2w_1 + \theta^2 w_1 + \theta w_2}{6-9\theta^2+2\theta^4}.
\end{aligned}$$

Substituting $p_i(w_1, w_2)$ and $e_1(w_1, w_2)$ into the manufacturers' profit functions yields

$$\begin{aligned}
\Pi_{RN-m1}(w_1) &= \frac{2w_1\left(\left(2-\theta^2\right)A_1 - \theta A_2 - 2w_1 + \theta^2 w_1 + \theta w_2\right)}{6-9\theta^2+2\theta^4}; \\
\Pi_{RN-m2}(w_2) &= \frac{w_2\left(-2\theta A_1 + \left(3-2\theta^2\right)A_2 + 2\theta w_1 - 3w_2 + 2\theta^2 w_2\right)}{6-9\theta^2+2\theta^4}.
\end{aligned}$$

$\Pi_{RN-m1}(w_1)$ is concave in w_i because $\frac{\partial^2 \Pi_{RN-m1}}{\partial w_1^2} = -\frac{4\left(2-\theta^2\right)}{6-9\theta^2+2\theta^4} < 0$ as long as $\theta < \frac{\sqrt{9-\sqrt{33}}}{2}$, which holds in the feasible area. So, the unique equilibrium wholesale prices w_{RN-i}^* are as follows:

$$\begin{aligned}
w_{RN-1}^* &= \frac{4\left(3-4\theta^2+\theta^4\right)A_1 + \theta\left(-3+2\theta^2\right)A_2}{24-30\theta^2+8\theta^4}; \\
w_{RN-2}^* &= \frac{\theta\left(-2+\theta^2\right)A_1 + 2\left(3-4\theta^2+\theta^4\right)A_2}{12-15\theta^2+4\theta^4}.
\end{aligned}$$

The equilibrium wholesale prices lead to the equilibrium retail prices p_{RN-i}^* and advertising level e_{RN-1}^* that follow.

$$\begin{aligned}
p_{RN-1}^* &= \frac{\left(10-15\theta^2+4\theta^4\right)\left(4\left(3-4\theta^2+\theta^4\right)A_1 + \theta\left(-3+2\theta^2\right)A_2\right)}{2\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)}; \\
p_{RN-2}^* &= \frac{\left(9-14\theta^2+4\theta^4\right)\left(\theta\left(-2+\theta^2\right)A_1 + 2\left(3-4\theta^2+\theta^4\right)A_2\right)}{72-198\theta^2+183\theta^4-66\theta^6+8\theta^8};
\end{aligned}$$

$$e^*_{RN-1} = \frac{(2-\theta^2)\left(4\left(3-4\theta^2+\theta^4\right)A_1 - \theta\left(3-2\theta^2\right)A_2\right)}{2\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)}.$$

The equilibrium profits and demands for the retailers are given by

$$\begin{aligned}
\Pi^*_{RN-r1} &= \frac{\left(2-\theta^2\right)^2\left(3-4\theta^2\right)\left(4\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(3-2\theta^2\right)A_2\right)^2}{4\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)^2}, \\
\Pi^*_{RN-r2} &= \frac{\left(3-2\theta^2\right)^2\left(1-\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)^2}{\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)^2}, \\
D^*_{RN-1} &= \frac{\left(2-\theta^2\right)\left(4\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(3-2\theta^2\right)A_2\right)}{72-198\theta^2+183\theta^4-66\theta^6+8\theta^8}, \\
D^*_{RN-2} &= \frac{\left(3-2\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)}{72-198\theta^2+183\theta^4-66\theta^6+8\theta^8}.
\end{aligned}$$

For these prices and demands to be nonnegative requires

$$4(3-4\theta^2+\theta^4)A_1 \geq \theta(3-2\theta^2)A_2 \quad \text{and} \quad 2(3-4\theta^2+\theta^4)A_2 \geq \theta(2-\theta^2)A_1,$$

which is equivalent to $\frac{\theta(3-2\theta^2)}{4(3-4\theta^2+\theta^4)} \leq \Omega \leq \frac{2(3-4\theta^2+\theta^4)}{\theta(2-\theta^2)}$. The largest feasible domain for θ is $[0, 0.902]$, as the upper bound of θ is obtained when the above two constraint boundaries cross.

In NR, symmetrically, the equilibrium for the retailers is given by

$$\begin{aligned}
\Pi^*_{NR-r1} &= \frac{\left(3-2\theta^2\right)^2\left(1-\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)^2}, \\
\Pi^*_{NR-r2} &= \frac{\left(2-\theta^2\right)^2\left(3-4\theta^2\right)\left(4\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(3-2\theta^2\right)A_1\right)^2}{4\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)^2}, \\
D^*_{NR-1} &= \frac{\left(3-2\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)}{72-198\theta^2+183\theta^4-66\theta^6+8\theta^8}, \\
D^*_{NR-2} &= \frac{\left(2-\theta^2\right)\left(4\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(3-2\theta^2\right)A_1\right)}{72-198\theta^2+183\theta^4-66\theta^6+8\theta^8}.
\end{aligned}$$

For the prices and demands to be nonnegative requires

$$2(3-4\theta^2+\theta^4)A_1 \geq \theta(2-\theta^2)A_2 \quad \text{and} \quad 4(3-4\theta^2+\theta^4)A_2 \geq \theta(3-2\theta^2)A_1,$$

where the largest feasible domain of θ is given by $\theta \in [0, 0.902]$ as the upper bound of θ is obtained when the above two constraint boundaries cross.

In the following, without loss of generality, we compare Retailer 1's profits in the various cases. To compare RN, NR, and NN, we can get their boundary values

$$\hat{\Omega}^{RN-NN}_{r1}(\theta) = \min\{\hat{\Omega}^{RN-NN}_{r1-1}(\theta), \bar{\Omega}^{RN-NN}(\theta)\},$$

$$\hat{\Omega}_{r1}^{NR-NN}(\theta) \quad = \quad \min\{\hat{\Omega}_{r1-1}^{NR-NN}(\theta), \bar{\Omega}^{NR-NN}(\theta)\},$$

where

$$\bar{\Omega}^{RN-NN}(\theta) = \frac{2\left(3-4\theta^2+\theta^4\right)}{\theta\left(2-\theta^2\right)} \quad \text{and} \quad \bar{\Omega}^{NR-NN}(\theta) = \frac{\theta(3-2\theta^2)}{4(3-4\theta^2+4\theta^4)},$$

which also ensure the nonnegative prices and demands for RN (NR). Meanwhile,

$$\hat{\Omega}_{r1-1}^{RN-NN}(\theta) \quad = \quad \frac{M_1 + M_2\sqrt{3-7\theta^2+4\theta^4}}{221184 - 1603584\theta^2 + M_3},$$

$$\hat{\Omega}_{r1-1}^{NR-NN}(\theta) \quad = \quad \frac{96 - 256\theta^2 + 270\theta^4 - 143\theta^6 + 38\theta^8 - 4\theta^{10}}{72\theta - 170\theta^3 + 142\theta^5 - 49\theta^7 + 6\theta^9},$$

where

$$M_1 = 552296\theta - 3571200\theta^3 + 10073280\theta^5 - 16355200\theta^7 + 16937420\theta^9 - 11694700\theta^{11} + 5452760\theta^{13} - 1694980\theta^{15} + 336120\theta^{17} - 38400\theta^{19} + 1920\theta^{21},$$

$$M_2 = \theta^3\left(4608 - 187200\theta^2 + 307200\theta^4 - 264180\theta^6 + 128870\theta^8 - 35820\theta^{10} + 5280\theta^{12} - 320\theta^{14}\right),$$

$$M_3 = 51217920\theta^4 - 95037440\theta^6 + 113735520\theta^8 - 92128800\theta^{10} + 51543660\theta^{12} - 19930080\theta^{14} + 5225680\theta^{16} - 886320\theta^{18} + 87680\theta^{20} - 3840\theta^{22}.$$

We have $\Pi^*_{RN-r1} > \Pi^*_{NN-r1}$ if and only if $\Omega > \hat{\Omega}_{r1}^{RN-NN}(\theta)$ and $\Pi^*_{RR-r1} > \Pi^*_{NR-r1}$ if and only if $\Omega > \hat{\Omega}_{r1}^{RR-NR}(\theta)$. Contour plots clearly demonstrate that $\hat{\Omega}_{r1}^{RN-NN}(\theta) < \hat{\Omega}_{r1}^{RR-NR}(\theta) < 1$, and that $\hat{\Omega}_{r1}^{RN-NN}(\theta)$ and $\hat{\Omega}_{r1}^{RR-NR}(\theta)$ increase with θ. [11] These contour plots, similar to Figure 1 and others in this paper, are unique because θ is in $[0,1)$, η is in $[0,1]$, and we need only consider Ω in $[0,1]$ (for cases where the base demands are not symmetric). When we cover these feasible domains, the function crosses the zero only once. We can provide any of the dozens of contour plots used in this paper, but omit them here to focus the exposition.

To compare RN, NR, and RR, we compute their boundary values as follows.

$$\hat{\Omega}_{r1}^{RR-RN}(\theta) \quad = \quad \min\{\hat{\Omega}_{r1-1}^{RR-RN}(\theta), \bar{\Omega}^{RR-RN}(\theta)\},$$

$$\hat{\Omega}_{r1}^{RR-NR}(\theta) \quad = \quad \min\{\hat{\Omega}_{r1-1}^{RR-NR}(\theta), \bar{\Omega}^{RR-NR}(\theta)\},$$

where

$$\bar{\Omega}^{RR-RN}(\theta) = \bar{\Omega}^{RR-NR}(\theta) = \frac{9-14\theta^2+4\theta^4}{\theta\left(3-2\theta^2\right)},$$

because $\frac{\theta\left(3-2\theta^2\right)}{(9-14\theta^2+4\theta^4)} > \frac{\theta\left(2-\theta^2\right)}{2(3-4\theta^2+\theta^4)} > \frac{\theta\left(3-2\theta^2\right)}{4(3-4\theta^2+\theta^4)}$, and

$$\hat{\Omega}_{r1-1}^{RR-RN}(\theta) \quad = \quad \frac{162 - 513\theta^2 + 642\theta^4 - 404\theta^6 + 128\theta^8 - 16\theta^{10}}{162\theta - 447\theta^3 + 434\theta^5 - 172\theta^7 + 24\theta^9},$$

[11] A contour line (also isoline or isarithm) of a function of two variables is a curve of all combinations of the two variables along which the function has a constant value (specifically zero in every one of our applications of this technique). For example, $\hat{\Omega}_{r1}^{RN-NN}(\theta)$ is a contour line of $\Pi^*_{RN-r1} - \Pi^*_{NN-r1} = 0$.

$$\hat{\Omega}_{r1-1}^{RR-NR}(\theta) = \frac{N_1 + 2N_2\sqrt{\left(1-\theta^2\right)^3\left(3-4\theta^2\right)}}{314928 - 2974320\theta^2 + N_3},$$

where

$$\begin{aligned} N_1 &= 104976\theta - 880632\theta^3 + 3297996\theta^5 - 7269156\theta^7 \\ &\quad + 104618849\theta^9 - 103060004\theta^{11} + 70781132\theta^{13} - 33827760\theta^{15} + 1100512\theta^{17} - 231776\theta^{19} + 28416\theta^{21} - 1536\theta^{23}, \\ N_2 &= \theta^3\left(-5832 + 28998\theta^2 - 57663\theta^4 + 59238\theta^6 - 33996\theta^8 + 10968\theta^{10} - 1856\theta^{12} + 128\theta^{14}\right), \\ N_3 &= 126214200\theta^4 - 317426040\theta^6 + 525630510\theta^8 - 602274360\theta^{10} \\ &\quad + 488572160\theta^{12} - 282244160\theta^{14} + 115121280\theta^{16} - 32323840\theta^{18} + 5933440\theta^{20} - 640000\theta^{22} + 3072\theta^{24}. \end{aligned}$$

So the common feasible area for θ is $\theta \in [0, 0.823]$ where the upper bound of θ is reached when the nonnegativity constraint lines cross and the domain will be narrower as Ω decreases. We have $\Pi^*_{NR-r1} < \Pi^*_{NN-r1}$ if and only if $\Omega < \hat{\Omega}_{r1}^{NR-NN}(\theta)$ and $\Pi^*_{RR-r1} < \Pi^*_{RN-r1}$ if and only if $\Omega < \hat{\Omega}_{r1}^{RR-RN}(\theta)$. Contour plots demonstrate that $1 < \hat{\Omega}_{r1}^{RR-RN}(\theta) < \hat{\Omega}_{r1}^{NR-NN}(\theta)$, and that $\hat{\Omega}_{r1}^{RR-RN}(\theta)$ and $\hat{\Omega}_{r1}^{NR-NN}(\theta)$ decrease with θ.

Similar methods yield the boundary values for Retailer 2 in NR, RN, and NN as follows.

$$\begin{aligned} \hat{\Omega}_{r2}^{RN-NN}(\theta) &= \min\{\hat{\Omega}_{r2-1}^{RN-NN}(\theta), \underline{\Omega}^{RN-NN}(\theta)\}, \\ \hat{\Omega}_{r2}^{NR-NN}(\theta) &= \min\{\hat{\Omega}_{r2-1}^{NR-NN}(\theta), \underline{\Omega}^{NR-NN}(\theta)\}, \end{aligned}$$

where

$$\begin{aligned} \underline{\Omega}^{RN-NN}(\theta) &= \frac{\theta\left(3-2\theta^2\right)}{4\left(3-4\theta^2+\theta^4\right)}, \\ \underline{\Omega}^{NR-NN}(\theta) &= \frac{\theta\left(2-\theta^2\right)}{2\left(3-4\theta^2+\theta^4\right)}, \\ \hat{\Omega}_{r2-1}^{RN-NN}(\theta) &= \frac{2304 - 10008\theta^2 + 17854\theta^4 - 16922\theta^6 + 9189\theta^8 - 2858\theta^{10} + 472\theta^{12} - 32\theta^{14}}{672\theta - 2416\theta^3 + 3462\theta^5 - 2531\theta^7 + 996\theta^9 - 200\theta^{11} + 16\theta^{13}}, \\ \hat{\Omega}_{r2-1}^{NR-NN}(\theta) &= \frac{2M_1 + M_2\sqrt{3-7\theta^2+4\theta^4}}{\theta^2 M_4}, \end{aligned}$$

where

$$M_4 = 27648 - 156672\theta^2 + 383856\theta^4 - 536296\theta^6 + 472531\theta^8 - 272667\theta^{10} + 102920\theta^{12} - 24428\theta^{14} + 3296\theta^{16} - 192\theta^{18}.$$

The boundaries for Retailer 2 in NR, RN, and RR are as follows.

$$\hat{\Omega}_{r2}^{RR-RN}(\theta) = \min\{\hat{\Omega}_{r2-1}^{RR-RN}(\theta), \underline{\Omega}^{RR-RN}(\theta)\},$$

$$\hat{\Omega}_{r2}^{RR-NR}(\theta) \quad = \quad \min\{\hat{\Omega}_{r2-1}^{RR-NR}(\theta), \underline{\Omega}^{RR-NR}(\theta)\},$$

where

$$\begin{aligned}
\underline{\Omega}^{RR-RN}(\theta) &= \underline{\Omega}^{RR-NR}(\theta) = \frac{\theta\left(3-2\theta^2\right)}{9-14\theta^2+4\theta^4}, \\
\hat{\Omega}_{r2-1}^{RR-RN}(\theta) &= \frac{N_1+2\sqrt{N_2}}{\theta^2 N_4}, \\
\hat{\Omega}_{r2-1}^{RR-NR}(\theta) &= \frac{3888-19494\theta^2+39849\theta^4-42950\theta^6+26332\theta^8-9192\theta^{10}+1696\theta^{12}-128\theta^{14}}{11340-47790\theta^3+79800\theta^5-67600\theta^7+30640\theta^9-7040\theta^{11}+640\theta^{13}},
\end{aligned}$$

where

$$N_4 = 34992-256608\theta^2+833976\theta^4-1584432\theta^6+1950599\theta^8-1625544\theta^{10}+926840\theta^{12}-355712\theta^{14}+87536\theta^{16}-12416\theta^{18}+768\theta^{20}.$$

We have $\Pi^*_{NR-r2} > \Pi^*_{NN-r2}$ if and only if $\Omega < \hat{\Omega}_{r2}^{NR-NN}(\theta)$ and $\Pi^*_{RR-r2} > \Pi^*_{RN-r2}$ if and only if $\Omega < \hat{\Omega}_{r2}^{RR-RN}(\theta)$. We can show that $\hat{\Omega}_{r2}^{NR-NN}(\theta) > \hat{\Omega}_{r2}^{RR-RN}(\theta) > 1$, and that $\hat{\Omega}_{r2}^{NR-NN}(\theta)$ and $\hat{\Omega}_{r2}^{RR-RN}(\theta)$ decrease with θ. Also, we have $\Pi^*_{RN-r2} < \Pi^*_{NN-r2}$ if and only if $\Omega > \hat{\Omega}_{r2}^{RN-NN}(\theta)$ and $\Pi^*_{RR-r2} < \Pi^*_{NR-r1}$ if and only if $\Omega > \hat{\Omega}_{r2}^{RR-NR}(\theta)$. We observe that $1 > \hat{\Omega}_{r2}^{RR-NR}(\theta) > \hat{\Omega}_{r2}^{RN-NN}(\theta)$, and that $\hat{\Omega}_{r2}^{RR-NR}(\theta)$ and $\hat{\Omega}_{r2}^{RN-NN}(\theta)$ increase with θ. □

Proof of Theorem 2: Consider RN. As argued in Lemma 2, Retailer 2 prefers RN to RR as long as $\Omega > \hat{\Omega}_{r2}^{RR-RN}(\theta)$. Meanwhile, Retailer 1 prefers RN to NN as long as $\Omega > \hat{\Omega}_{r1}^{RN-NN}(\theta)$, where $\hat{\Omega}_{r1}^{RN-NN}(\theta) < \hat{\Omega}_{r2}^{RR-RN}(\theta)$. Thus neither retailer would deviate from RN as long as $\hat{\Omega}_{r2}^{RR-RN}(\theta) < \Omega < \bar{\Omega}^{RR}(\theta)$. NR also has this property by symmetry. By a similar argument, Retailer 2 prefers RR to RN as long as $\Omega < \hat{\Omega}_{r2}^{RR-RN}(\theta)$ and Retailer 1 prefers RR to NR as long as $\Omega > \hat{\Omega}_{r1}^{RR-NR}(\theta)$. Thus RR is an equilibrium if and only if $\hat{\Omega}_{r1}^{RR-NR}(\theta) < \Omega < \hat{\Omega}_{r2}^{RR-RN}(\theta)$. Worth noting is that at least one player could perform better in NN than in RR. However, this occurs outside the common feasible domain for $\theta \in [0, 0.823]$ so falls beyond the scope of our discussion. □

Proof of Lemma 3: Manufacturer advertising with cost sharing presents four possible outcomes: CSMM, CSMN, CSNM, and CSNN. The profit functions of CSMM are documented in Eq. (7) and those of CSMN and CSNM can be inferred similarly given that only one manufacturer advertises. CSNN is equivalent to NN (since cost sharing has no impact when no parties advertise), which was analyzed earlier. For brevity, below we list only equilibrium solutions for the symmetric setting ($A_1 = A_2 = 1$).

The equilibrium actions and outcomes for Manufacturer i and Retailer i in various cases are as follows. In CSMM:

$$
\begin{aligned}
w_{CSMM-i} &= \frac{2(1-\eta)\left(4-5\theta^2+\theta^4\right)}{14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)};\\
p_{CSMM-i} &= \frac{4(1-\eta)\left(3-4\theta^2+\theta^4\right)}{14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)};\\
e_{CSMM-mi} &= \frac{2-\theta^2}{14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)};\\
D_{CSMM-i} &= \frac{2(1-\eta)\left(2-\theta^2\right)}{14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)};\\
\Pi_{CSMM-mi} &= \frac{(1-\eta)\left(2-\theta^2\right)\left(14-19\theta^2+4\theta^4-4\eta\left(4-5\theta^2+\theta^4\right)\right)}{\left(14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)\right)^2};\\
\Pi_{CSMM-ri} &= \frac{\left(2-\theta^2\right)^2\left(4-9\eta+4\eta^2-4(1-\eta)^2\theta^2\right)}{\left(14-2\eta(2-\theta)(1+\theta)\left(4-\theta-2\theta^2\right)+\theta\left(4-\theta\left(17+2\theta-4\theta^2\right)\right)\right)^2}.
\end{aligned}
$$

In CSMN:

$$
\begin{aligned}
w_{CSMN-i} &= \frac{2(1-\eta)(2-\theta)(1+\theta)\left(2-\theta-\theta^2\right)^2(4+\theta(1-2\theta))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
p_{CSMN-i} &= \frac{4(1-\eta)(1-\theta)^2(1+\theta)(2+\theta)\left(3-\theta^2\right)(4+\theta(1-2\theta))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
e_{CSMN-m1} &= \frac{\left(2-\theta^2\right)\left(2-\theta-\theta^2\right)(4+\theta(1-2\theta))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
D_{CSMN-i} &= \frac{2(1-\eta)(1-\theta)(2+\theta)\left(2-\theta^2\right)(4+\theta(1-2\theta))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
\Pi_{CSMN-mi} &= \frac{(1-\eta)(1-\theta)^2(2+\theta)^2\left(2-\theta^2\right)(4+\theta(1-2\theta))^2\left(14-19\theta^2+4\theta^4-4\eta\left(4-5\theta^2+\theta^4\right)\right)}{\left(16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8\right)^2};\\
\Pi_{CSMN-ri} &= \frac{(1-\theta)^2(2+\theta)^2\left(2-\theta^2\right)^2\left(4-9\eta+4\eta^2-4(1-\eta)^2\theta^2\right)(4+\theta(1-2\theta))^2}{\left(16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8\right)^2}.
\end{aligned}
$$

In CSNM:

$$
\begin{aligned}
w_{CSNM-i} &= \frac{\left(4-5\theta^2+\theta^4\right)(14-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta(4+\theta(17-2\theta(1+2\theta))))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
p_{CSNM-i} &= \frac{2\left(3-4\theta^2+\theta^4\right)(14-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta(4+\theta(17-2\theta(1+2\theta))))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
e_{CSNM-m2} &= \frac{\left(2-\theta^2\right)\left(2-\theta-\theta^2\right)(4+\theta(1-2\theta))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};\\
D_{CSNM-i} &= \frac{\left(2-\theta^2\right)(14-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta(4+\theta(17-2\theta(1+2\theta))))}{16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8};
\end{aligned}
$$

$$\Pi_{CSNM-mi} = \frac{\left(4-\theta^2\right)\left(2-3\theta^2+\theta^4\right)\left(14-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta(4+\theta(17-2\theta(1+2\theta)))\right)^2}{\left(16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8\right)^2};$$

$$\Pi_{CSNM-ri} = \frac{\left(2-\theta^2\right)^2\left(1-\theta^2\right)\left(14-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta(4+\theta(17-2\theta(1+2\theta)))\right)^2}{\left(16(7-8\eta)-2(135-148\eta)\theta^2+13(17-18\eta)\theta^4-2(36-37\eta)\theta^6+8(1-\eta)\theta^8\right)^2}.$$

The equilibria for the rival manufacturer and retailer follow by symmetry. For example, for Manufacturer 2 in CSMN, $w_{CSMN-1} = w_{CSNM-2}$ and $p_{CSMN-1} = p_{CSNM-2}$. To ensure a meaningful comparison, we enforce the common feasible domain for all cases. That is, $\eta < \hat{\eta}_{mi}^{CSMM}(\theta) \equiv \frac{14-4\theta-17\theta^2+2\theta^3+4\theta^4}{2(8-2\theta-9\theta^2+\theta^3+2\theta^4)}$.

We first compare CSMN and CSNN (i.e., NN). Define $\Delta\Pi_{m1}^{CSMN-NN} \equiv \Pi_{CSMN-m1} - \Pi_{NN-m1}$ as Manufacturer 1's profit in CSMN minus that in NN. This is strictly positive if and only if $\eta < \hat{\eta}_{m1}^{CSMN-NN}(\theta) \equiv \frac{896-3232\theta^2+4570\theta^4-3222\theta^6+1191\theta^8-220\theta^{10}+16\theta^{12}}{1024-3584\theta^2+4940\theta^4-3407\theta^6+1235\theta^8-224\theta^{10}+16\theta^{12}}$, which exceeds $\eta_{m1}^{CSMM}(\theta)$ and thus lies outside the common feasible domain. Hence, $\Pi_{CSMN-m1} > \Pi_{NN-m1}$ throughout the common feasible domain. Next define $\Delta\Pi_{m1}^{CSMM-CSNM} \equiv \Pi_{CSMM-m1} - \Pi_{CSNM-m1}$. By contour plotting, we find $\Delta\Pi_{m1}^{CSMM-CSNM} > 0$ for any θ and η in the feasible domain. Therefore, Manufacturer 1 always benefits from its own advertising under cost sharing at any cost sharing rate. So does Manufacturer 2. Thus, CSMM is the unique equilibrium for manufacturer advertising with cost sharing given $\Omega = 1$.

We now compare CSMM with NN. Define $\Delta\Pi_{mi}^{CSMM-NN} \equiv \Pi_{CSMM-mi} - \Pi_{NN-mi}$. This is strictly positive if and only if $\eta < \hat{\eta}_{mi}^{CSMM-NN}(\theta) \equiv \frac{2\left(14-7\theta-27\theta^2+9\theta^3+14\theta^4-2\theta^5-2\theta^6\right)}{32-16\theta-58\theta^2+19\theta^3+29\theta^4-4\theta^5-4\theta^6}$, which is outside the common feasible domain when $\theta < 0.676$. Therefore, $\Pi_{CSMM-mi} < \Pi_{NN-mi}$ when $\eta > \hat{\eta}_{mi}^{CSMM-NN}(\theta)$; otherwise, $\Pi_{CSMM-mi} \geq \Pi_{NN-mi}$. Figure 13 illustrates this property, which implies that the manufacturers might encounter a Prisoner's Dilemma under manufacturer advertising with cost sharing.□

Proof of Lemma 4: Retailer advertising with cost sharing presents four possible outcomes: CSRR, CSRN, CSNR, and CSNN. The profit functions under CSRR are documented in Eq. (6) and those of CSRN and CSNR can be inferred similarly given that only one retailer advertises. Again, CSNN is equivalent to Case NN. For brevity, we present findings only for the symmetric setting of $A_1 = A_2 = 1$.

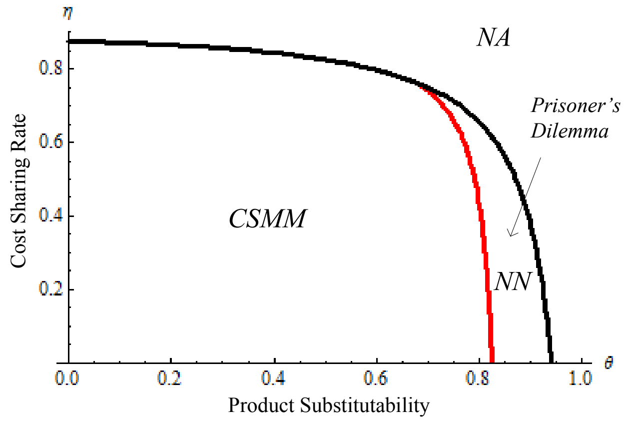

Figure 13: Manufacturer 1's preference between CSMM and NN, given $\Omega = 1$

The equilibrium actions and outcomes for Manufacturer i and Retailer i in various cases are as follows.

In CSRR:

$$
\begin{aligned}
w_{CSRR-i} &= \frac{9-30\eta+36\eta^2-16\eta^3-2(1-\eta)(8-\eta(17-10\eta))\theta^2+4(1-\eta)^3\theta^4}{18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4}; \\
p_{CSRR-i} &= \frac{15-50\eta+58\eta^2-24\eta^3-2(1-\eta)(13-4\eta(7-4\eta))\theta^2+8(1-\eta)^3\theta^4}{18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4}; \\
e_{CSRR-ri} &= \frac{(1-\eta)\left(3-4\eta-2(1-\eta)\theta^2\right)}{18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4}; \\
D_{CSRR-i} &= \frac{2(1-\eta)^2\left(3-4\eta-2(1-\eta)\theta^2\right)}{18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4}; \\
\Pi_{CSRR-mi} &= \frac{(1-\eta)^2\left((3-4\eta)^2(6-\eta(13-8\eta))-4(1-\eta)(3-4\eta)(11-2\eta(12-7\eta))\theta^2+4(1-\eta)^2(22-7\eta(7-4\eta))\theta^4-16(1-\eta)^4\theta^6\right)}{\left(18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4\right)^2}; \\
\Pi_{CSRR-ri} &= \frac{(1-\eta)^3\left(3-4\eta-2(1-\eta)\theta^2\right)^2\left(3-4\eta-4(1-\eta)\theta^2\right)}{\left(18-63\eta+76\eta^2-32\eta^3+2(1-\eta)^2(3-4\eta)\theta-2(1-\eta)(14-\eta(31-18\eta))\theta^2-4(1-\eta)^3\theta^3+8(1-\eta)^3\theta^4\right)^2}.
\end{aligned}
$$

In CSRN:

$$
\begin{aligned}
w_{CSRN-1} &= \frac{\left(6-4\eta(3-2\eta)-(9-2(9-5\eta)\eta)\theta^2+2(1-\eta)^2\theta^4\right)CS_3}{CS_1}; \\
p_{CSRN-1} &= \frac{\left(2(5-2\eta(5-3\eta))-(15-2(15-8\eta)\eta)\theta^2+4(1-\eta)^2\theta^4\right)CS_3}{CS_1}; \\
e_{CSRN-r1} &= \frac{(1-\eta)\left(2-\theta^2\right)CS_3}{CS_1}; \\
D_{CSRN-1} &= \frac{(1-\eta)^2\left(2-\theta^2\right)CS_3}{CS_2}; \\
\Pi_{CSRN-m1} &= \frac{(1-\eta)^2\left(2-\theta^2\right)\left(2(6-\eta(13-8\eta))-(18-(37-20\eta)\eta)\theta^2+4(1-\eta)^2\theta^4\right)CS_3^2}{4CS_2^2}; \\
\Pi_{CSRN-r1} &= \frac{(1-\eta)^3\left(2-\theta^2\right)^2\left(3-4\eta-4(1-\eta)\theta^2\right)CS_3^2}{4CS_2^2}.
\end{aligned}
$$

In CSNR:

$$
\begin{aligned}
w_{CSNR-1} &= \frac{\left(6-9\theta^2+2\theta^4-2\eta\left(4-5\theta^2+\theta^4\right)\right)CS_4}{CS_1};\\
p_{CSNR-1} &= \frac{\left(9-12\eta-2(7-8\eta)\theta^2+4(1-\eta)\theta^4\right)CS_4}{CS_1};\\
D_{CSNR-1} &= \frac{\left(3-4\eta-2(1-\eta)\theta^2\right)CS_4}{CS_1};\\
\Pi_{CSNR-m1} &= \frac{\left(3-4\eta-2(1-\eta)\theta^2\right)\left(6-9\theta^2+2\theta^4-2\eta\left(4-5\theta^2+\theta^4\right)\right)CS_4^2}{4CS_2^2};\\
\Pi_{CSNR-r1} &= \frac{\left(1-\theta^2\right)\left(3-4\eta-2(1-\eta)\theta^2\right)^2 CS_4^2}{4CS_2^2}.
\end{aligned}
$$

In the above,

$$
\begin{aligned}
CS_1 &= 8(3-4\eta)(6-\eta(13-8\eta))-4(99-\eta(331-2(189-74\eta)\eta))\theta^2\\
&\quad +2(183-2\eta(293-3\eta(106-39\eta)))\theta^4-4(1-\eta)(33-\eta(69-37\eta))\theta^6+16(1-\eta)^3\theta^8;\\
CS_2 &= 4(3-4\eta)(6-\eta(13-8\eta))-2(99-\eta(331-2(189-74\eta)\eta))\theta^2\\
&\quad +(183-2\eta(293-3\eta(106-39\eta)))\theta^4-2(1-\eta)(33-\eta(69-37\eta))\theta^6+8(1-\eta)^3\theta^8;\\
CS_3 &= 12-2\eta(1-\theta)(2+\theta)(4+\theta(1-2\theta))-\theta\left(3+2\theta\left(8-\theta-2\theta^2\right)\right);\\
CS_4 &= 2(6-\eta(13-8\eta))-4(1-\eta)^2\theta-(16-3(11-6\eta)\eta)\theta^2+2(1-\eta)^2\theta^3+4(1-\eta)^2\theta^4.
\end{aligned}
$$

The equilibrium actions and outcomes for the other manufacturer and retailer in each setting can be easily obtained by symmetry. For example, $w_{CSRN-1}=w_{CSNR-2}$ and $p_{CSRN-1}=p_{CSNR-2}$. The common feasible area for all forms of retailer advertising with cost sharing is $\eta<\frac{3-2\theta^2}{2(2-\theta^2)}\equiv\hat{\eta}_{ri}^{CSRR}(\theta)$.

We now compare CSRN and CSNN. Define $\Delta\Pi_{r1}^{CSRN-NN}\equiv\Pi_{CSRR-r1}-\Pi_{NN-r1}$ as Retailer 1's profits in CSRN minus the one in NN. We prove the existence of $\hat{\eta}_{r1}^{CSRN-NN}(\theta)$ by characterizing $\Delta\Pi_{r1}^{CSRN-NN}=0$ through a contour plot. The threshold curve is then uniquely represented by $\hat{\eta}_{r1}^{CSRN-NN}(\theta)$, because there are only two viable parameters. $\hat{\eta}_{r1}^{CSRN-NN}(\theta)$ is in the middle of the feasible domain and decreases with θ.

Note that $\hat{\eta}_{r1}^{CSRN-NN}$ is equivalent to $\hat{\eta}_{r2}^{CSNR-NN}$ by symmetry. Therefore, no retailer will unilaterally deviate from NN if and only if $\eta>\hat{\eta}_{r1}^{CSRN-NN}$.

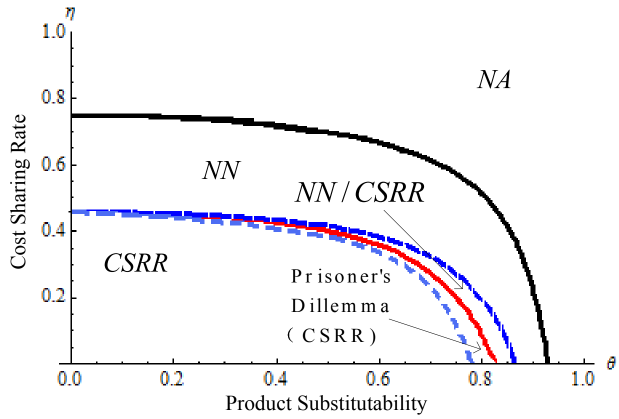

Figure 14: Equilibrium analysis in retailer advertising with cost sharing, given $\Omega = 1$.

Now compare CSRR and CSRN. Define $\Delta\Pi_{r2}^{CSRR-CSRN} \equiv \Pi_{CSRR-r2} - \Pi_{CSRN-r2}$. We obtain $\hat{\eta}_{r2}^{CSRR-CSRN}(\theta)$ from the contour plot of $\Delta\Pi_{r2}^{CSRR-CSRN} = 0$. $\hat{\eta}_{r2}^{CSRR-CSRN}(\theta)$ is in the middle of feasible domain and decreases with θ.

Note that $\hat{\eta}_{r1}^{CSRR-CSNR}$ is equivalent to $\hat{\eta}_{r2}^{CSRR-CSRN}$. Therefore, no retailer will unilaterally deviate from CSRR if and only if $\eta < \hat{\eta}_{r1}^{CSRR-CSNR}$. It is worth noting that $\eta_{r1}^{CSRN-NN} < \hat{\eta}_{r1}^{CSRR-CSNR}$. That is, a domain exists in which both CSRR and NN can be equilibria, as illustrated in Figure 14.

We now compare CSRR and NN. Define $\Delta\Pi_{r1}^{CSRR-NN} \equiv \Pi_{CSRR-r1} - \Pi_{NN-r1}$. By contour plotting we obtain a unique $\hat{\eta}^{CSRR-NN}(\theta)$ from $\Delta\Pi_{r1}^{CSRR-NN} = 0$. Since $\hat{\eta}^{CSRR-NN}(\theta) < \hat{\eta}_{r1}^{CSRN-NN} < \hat{\eta}_{r1}^{CSRR-CSNR}$, the retailers encounter a Prisoner's Dilemma when $\hat{\eta}_{r1}^{CSRR-NN}(\theta) < \eta < \hat{\eta}_{r1}^{CSRN-NN}(\theta)$, because both retailers are harmed by their advertising even though advertising is a dominant equilibrium strategy. Figure 14 summarizes all the above findings. □

Proof of Theorem 3: Because of symmetry the following proof needs only to consider Manufacturer 1 and Retailer 1. We first compare Manufacturer 1's profits between CSMM and MM and between CSRR and RR. Contour plotting shows that Manufacturer 1 prefers CSMM to MM as long as $\eta < \hat{\eta}_{m1}^{CSMM-MM}(\theta)$, where

$$\hat{\eta}_{m1}^{CSMM-MM}(\theta) = \frac{196 - 548\theta^2 - 16\theta^3 + 485\theta^4 + 8\theta^5 - 156\theta^6 + 16\theta^8}{8\left(28 - 75\theta^2 - 2\theta^3 + 64\theta^4 + \theta^5 - 20\theta^6 + 2\theta^8\right)}.$$

When $\hat{\eta}_{m1}^{CSMM-MM}(\theta) < \eta < \hat{\eta}_{mi}^{CSMM}(\theta)$, which is the common feasible area for all cases of manufacturer advertising with cost sharing, Manufacturer 1 prefers MM to CSMM. Similarly, Manu-

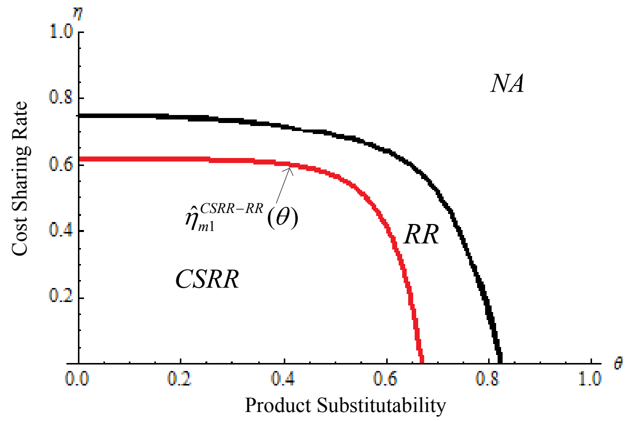

Figure 15: Manufacturer 1's preference between RR and CSRR given $\Omega = 1$.

facturer 1 prefers CSRR to RR as long as $\eta < \hat{\eta}_{m1}^{CSRR-RR}(\theta)$, where $\hat{\eta}_{m1}^{CSRR-RR}(\theta)$ is illustrated in Figure 15. If $\hat{\eta}_{m1}^{CSRR-RR}(\theta) < \eta < \hat{\eta}_{ri}^{CSRR}(\theta)$, which is the common feasible area for all cases of retailer advertising with cost sharing, Manufacturer 1 prefers RR to CSRR.

Now consider Retailer 1's profit differences between CSMM and MM and between CSRR and RR. Methods similar to those described earlier show that Retailer 1 prefers CSMM to MM as long as $\eta < \hat{\eta}_{r1}^{CSMM-MM}(\theta)$, where

$$\hat{\eta}_{r1}^{CSMM-MM}(\theta) = \frac{28 - 48\theta - 148\theta^2 + 64\theta^3 + 199\theta^4 - 20\theta^5 - 100\theta^6 + 16\theta^8}{4\left(60 + 16\theta - 160\theta^2 - 32\theta^3 + 151\theta^4 + 20\theta^5 - 59\theta^6 - 4\theta^7 + 8\theta^8\right)}.$$

If $\hat{\eta}_{r1}^{CSMM-MM}(\theta) < \eta < \hat{\eta}_{mi}^{CSMM}(\theta)$, Retailer 1 prefers MM to CSMM. We have $\hat{\eta}_{r1}^{CSMM-MM}(\theta) < \hat{\eta}_{m1}^{CSMM-MM}(\theta) < \hat{\eta}_{m1}^{CSRR-RR}(\theta)$. The contour plot on the θ,η plane shows that RR dominates CSRR for Retailer 1 throughout the entire feasible domain. □

Proof of Theorem 4: We explicitly present the proof for $U_{RR} > U_{MM}$ only. The others are similar in nature so we omit due to their length. They are available on request.

Consumer welfare (U, with superscripts and subscripts following the conventions used throughout this paper) is based on the utility of the representative consumer in Eq. (3). Some algebra yields

$$\begin{aligned} \triangle U^{RR-MM}(\theta) &\equiv U_{RR} - U_{MM} \\ &= \frac{\left(6 - 5\theta^2 + 2\theta^4\right)\left(n_1 \times \Omega^2 + n_2 \times 2\theta\Omega + n_3\right) A_2^2}{2\left(196 - 492\theta^2 + 417\theta^4 - 140\theta^6 + 16\theta^8\right)^2 \left(81 - 261\theta^2 + 280\theta^4 - 116\theta^6 + 16\theta^8\right)^2}, \end{aligned}$$

where

$$n_1 = 1238328 - 102024336\theta^2 + 372154338\theta^4 - 791899947\theta^6 + 1090522231\theta^8 - 1019350866\theta^{10} + 659563340\theta^{12} - 295403328\theta^{14}$$

$$
\begin{aligned}
& +89787200\theta^{16} - 17652480\theta^{18} + 2022400\theta^{20} - 102400\theta^{22}; \\
n_2 \quad = \quad & 86184 - 1271628\theta^2 + 6466026\theta^4 - 171754090\theta^6 + 277283410\theta^8 - 291814020\theta^{10} + 206752520\theta^{12} - 99406640\theta^{14} \\
& +31972480\theta^{16} - 6581760\theta^{18} + 783360\theta^{20} - 40960\theta^{22}; \\
n_3 \quad = \quad & 1238328 - 10202436\theta^2 + 37215438\theta^4 - 791899470\theta^6 + 1090522310\theta^8 - 1019350860\theta^{10} + 659563400\theta^{12} - 295403280\theta^{14} \\
& +89787200\theta^{16} - 17652480\theta^{18} + 2022400\theta^{20} - 102400\theta^{22}.
\end{aligned}
$$

In the common feasible domain, $\triangle U^{RR-MM}(\theta,\Omega)$ is convex with respect to Ω, and increases with Ω for $\Omega > 0$. Furthermore, $\triangle U^{RR-MM}(\theta,\Omega) > \triangle U^{RR-MM}(\theta,0) = \frac{\left(6-5\theta^2+2\theta^4\right)n_1}{2(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8)^2(81-261\theta^2+280\theta^4-11}$ which is positive in the common feasible domain. Hence $U_{RR} > U_{MM}$. $\square$

Proof of Corollary 1: The following discussion is based on the common feasible domain under both manufacturer and retailer advertising; that is, $\Omega \in [\frac{\theta(3-2\theta^2)}{9-14\theta^2+4\theta^4}, \frac{9-14\theta^2+4\theta^4}{\theta(3-2\theta^2)}]$ and $\theta \in [0, 0.823]$. Without loss of generality, we let $A_{3-i} = 1$ and then $\Omega = A_i$. For supply chain 1's advertising level, we consider the relationship between e and Ω for MM:

$$\frac{\partial e^*_{MM-m1}}{\partial \Omega} = \frac{(2-\theta^2)(14-17\theta^2+4\theta^4)}{196-492\theta^2+417\theta^4-140\theta^6+16\theta^8} > 0.$$

For RR,

$$\frac{\partial e^*_{RR-r1}}{\partial \Omega} = \frac{(3-2\theta^2)(9-14\theta^2+4\theta^4)}{162-522\theta^2+560\theta^4-232\theta^6+32\theta^8},$$

which is positive in the feasible domain. So, supply chain 1's advertising effort level increases with Ω. For supply chain 2's advertising level, we consider the relationship between e and Ω for MM:

$$\frac{\partial e^*_{MM-m2}}{\partial \Omega} = -\frac{2\theta\left(2-\theta^2\right)^2}{196-492\theta^2+417\theta^4-140\theta^6+16\theta^8} < 0.$$

For RR,

$$\frac{\partial e^*_{RR-r2}}{\partial \Omega} = -\frac{\theta\left(3-2\theta^2\right)^2}{162-522\theta^2+560\theta^4-232\theta^6+32\theta^8} < 0,$$

Similar results arise in the other subgames, which are omitted here for brevity.

Now consider the relationship between e and θ. For MM,

$$\frac{\partial e^*_{MM-m1}}{\partial \theta} = \frac{2\begin{pmatrix} -784+384\theta^2+2056\theta^4-2992\theta^6+1711\theta^8-460\theta^{10}+48\theta^{12} \\ +4\theta\left(1092-3388\theta^2+4281\theta^4-2824\theta^6+1034\theta^8-200\theta^{10}+16\theta^{12}\right)A_1 \end{pmatrix}}{\left(196-492\theta^2+417\theta^4-140\theta^6+16\theta^8\right)^2},$$

which is positive if and only if $A_1 > \frac{784-384\theta^2-2056\theta^4+2992\theta^6-1711\theta^8+460\theta^{10}-48\theta^{12}}{4\theta(1092-3388\theta^2+4281\theta^4-2824\theta^6+1034\theta^8-200\theta^{10}+16\theta^{12})} \doteq \Omega^{e-\theta}_{MM}$. For RR,

$$\frac{\partial e^*_{RR-r1}}{\partial \theta}=\frac{\begin{array}{c}-729+567\theta^2+2808\theta^4-5448\theta^6+4064\theta^8-14240\theta^{10}+192\theta^{12}\\+2\theta\left(2187-8640\theta^2+13812\theta^4-11472\theta^6+5280\theta^8-1280\theta^{10}+128\theta^{12}\right)A_1\end{array}}{2\left(81-261\theta^2+280\theta^4-116\theta^6+16\theta^8\right)^2},$$

which is positive if and only if $A_1 > \frac{729-567\theta^2-2808\theta^4+5448\theta^6-4064\theta^8+14240\theta^{10}-192\theta^{12}}{2\theta(2187-8640\theta^2+13812\theta^4-11472\theta^6+5280\theta^8-1280\theta^{10}+128\theta^{12})} \doteq \Omega^{e-\theta}_{RR}$.

Similar results arise in the other subgames, which are omitted here for brevity. □

Proof of Corollary 2: The following discussion is based on the common feasible domain of CSMM and CSRR, that is $\eta < \frac{14-4\theta-17\theta^2+2\theta^3+4\theta^4}{2(8-2\theta-9\theta^2+\theta^3+2\theta^4)} = \hat{\eta}^{CSMM}_{mi}(\theta)$. For CSMM,

$$\frac{\partial e_{CSMM-mi}}{\partial \theta} = \frac{2(1-\eta)\left(-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5\right)}{\left(14+4\theta-17\theta^2-2\theta^3+4\theta^4-2\eta\left(8+2\theta-9\theta^2-\theta^3+2\theta^4\right)\right)^2},$$

which is positive if and only if $-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5 > 0$. We define $\theta_{CSMM} \doteq arg\{\theta|-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5 = 0\}$, which is unique in the feasible domain. For CSRR,

$$\frac{\partial e_{CSRR-ri}}{\partial \theta} = \frac{2(1-\eta)^3\begin{pmatrix}-9+48\theta+12\theta^2-48\theta^3-4\theta^4+16\theta^5\\+4\eta^2\left(-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5\right)\\-4\eta\left(-6+31\theta+7\theta^2-28\theta^3-2\theta^4+8\theta^5\right)\end{pmatrix}}{\begin{pmatrix}-18-6\theta+28\theta^2+4\theta^3-8\theta^4+4\eta^3\left(8+2\theta-9\theta^2-\theta^3+2\theta^4\right)\\-2\eta^2\left(38+11\theta-49\theta^2-6\theta^3+12\theta^4\right)+\eta\left(63+20\theta-90\theta^2-12\theta^3+24\theta^4\right)\end{pmatrix}^2},$$

which is positive as long as $\eta > \frac{-6+31\theta+7\theta^2-28\theta^3-2\theta^4+8\theta^5+\sqrt{\theta^2+2\theta^3-8\theta^4}}{2(-4+20\theta+4\theta^2-16\theta^3-\theta^4+4\theta^5)} \doteq \eta_{CSRR}$. □

Proof of Theorem 5:

We provide results for Manufacturer 1 and Retailer 1 here, and invoke symmetry for Manufacturer 2 and Retailer 2. The following lemma is needed to prove Theorem 5.

Lemma 6 *Consider Manufacturer* 1 *and Retailer* 1 *in a scenario of hybrid advertising. Boundary values exist such that*

1. *Both Manufacturer* 1 *and Retailer* 1 *simultaneously prefer RM to MM if and only if* $\Omega >$

$\hat{\Omega}_{r1}^{RM-MM}(\theta)$*, but prefer MM to RM if and only if* $\Omega < \hat{\Omega}_{m1}^{RM-MM}(\theta)$*, where* $\hat{\Omega}_{r1}^{RM-MM}(\theta) > \hat{\Omega}_{m1}^{RM-MM}(\theta)$.

2. *Both Manufacturer* 1 *and Retailer* 1 *simultaneously prefer MR to RR if and only if* $\Omega < \min\{\hat{\Omega}_{r1-2}^{MR-RR}(\theta),$
$\hat{\Omega}_{m1}^{MR-RR}(\theta)\}$*, but prefer RR to MR if and only if* $\Omega > \max\{\hat{\Omega}_{r1-2}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\}$.

3. *Both Manufacturer* 1 *and Retailer* 1 *simultaneously prefer RM to NM if and only if* $\Omega > \hat{\Omega}_{r1}^{RM-NM}(\theta)$*, but prefer NM to RM if and only if* $\Omega < \hat{\Omega}_{m1}^{RM-NM}(\theta)$.

4. *Both Manufacturer* 1 *and Retailer* 1 *always prefer MR to NR.*

Proof of Lemma 6: We first follow the itemized sequence of results in Lemma 6 and then extend our proof to Manufacturer 2 and Retailer 2.

(1) *Compare MM to RM and MR.* We directly start with the unique equilibrium of RM that follows.

$$
\begin{aligned}
w_{RM-1}^{*} &= \frac{(6-9\theta^2+2\theta^4)((21-30\theta^2+8\theta^4)A_1-2\theta(3-2\theta^2)A_2)}{4(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8)},\\
w_{RM-2}^{*} &= \frac{(6-9\theta^2+2\theta^4)(2(3-4\theta^2+\theta^4)A_2-\theta(2-\theta^2)A_1)}{63-180\theta^2+172\theta^4-64\theta^6+8\theta^8},\\
p_{RM-1}^{*} &= \frac{(10-15\theta^2+4\theta^4)\left((21-30\theta^2+8\theta^4)A_1-2\theta\left(3-2\theta^2\right)A_2\right)}{4\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)},\\
p_{RM-2}^{*} &= \frac{(9-14\theta^2+4\theta^4)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)}{63-180\theta^2+172\theta^4-64\theta^6+8\theta^8},\\
e_{RM-r1}^{*} &= \frac{(2-\theta^2)\left((21-30\theta^2+8\theta^4)A_1-2\theta\left(3-2\theta^2\right)A_2\right)}{4\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)},\\
e_{RM-m2}^{*} &= \frac{(3-2\theta^2)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)}{2\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)},\\
D_{RM-1}^{*} &= \frac{(2-\theta^2)\left((21-30\theta^2+8\theta^4)A_1-2\theta\left(3-2\theta^2\right)A_2\right)}{2\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)},\\
D_{RM-2}^{*} &= \frac{(3-2\theta^2)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)}{63-180\theta^2+172\theta^4-64\theta^6+8\theta^8},\\
\Pi_{RM-r1}^{*} &= \frac{\left(2-\theta^2\right)^2\left(3-4\theta^2\right)\left((21-30\theta^2+8\theta^4)A_1-2\theta\left(3-2\theta^2\right)A_2\right)^2}{16\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2},\\
\Pi_{RM-r2}^{*} &= \frac{\left(3-2\theta^2\right)^2\left(1-\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_2-\theta\left(2-\theta^2\right)A_1\right)^2}{\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
\Pi^*_{RM-m1} &= \frac{(2-\theta^2)\,(6-9\theta^2+2\theta^4)\,((21-30\theta^2+8\theta^4)\,A_1-2\theta\,(3-2\theta^2)\,A_2)^{\,2}}{8\,(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8)^2},\\
\Pi^*_{RM-m2} &= \frac{(63-144\theta^2+92\theta^4-16\theta^6)\,(2\,(3-4\theta^2+\theta^4)\,A_2-\theta\,(2-\theta^2)\,A_1)^{\,2}}{4\,(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8)^2}.
\end{aligned}
$$

For prices and demands to be nonnegative requires

$$
(21-30\theta^2+8\theta^4)A_1-2\theta(3-2\theta^2)A_2\geq 0 \quad \text{and} \quad 2(3-4\theta^2+\theta^4)A_2-\theta(2-\theta^2)A_1\geq 0.
$$

Thus, the common lower and upper bounds for RM and MM are defined as follows:

$$
\underline{\Omega}^{RM-MM}(\theta)=\frac{2\theta\,(3-2\theta^2)}{21-30\theta^2+8\theta^4} \quad \text{and} \quad \bar{\Omega}^{RM-MM}(\theta)=\frac{2(3-4\theta^2+\theta^4)}{\theta(2-\theta^2)}.
$$

The feasible domain for θ is $\theta\in[0,0.876]$, where the upper bound of θ arises when the above two constraint lines cross, which is narrower than the domain of MM but wider than that of RR. Following the same steps as in the proof of Lemma 1, the boundary values of $\hat{\Omega}^{RM-MM}_{r1}(\theta)$ and $\hat{\Omega}^{RM-MM}_{m1}(\theta)$ result from equating the profits of RM and those of MM:

$$
\begin{aligned}
\hat{\Omega}^{RM-MM}_{r1}(\theta) &= \min\{\hat{\Omega}^{RM-MM}_{r1-1}(\theta),\bar{\Omega}^{RM-MM}(\theta)\},\\
\hat{\Omega}^{RM-MM}_{m1}(\theta) &= \min\{\hat{\Omega}^{RM-MM}_{m1-1}(\theta),\bar{\Omega}^{RM-MM}(\theta)\},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Omega}^{RM-MM}_{r1-1}(\theta) &= \frac{2(k_1+16k_2\sqrt{3-7\theta^2+4\theta^4})}{1037232-12940704\theta^2+k_3},\\
\hat{\Omega}^{RM-MM}_{m1-1}(\theta) &= \frac{2(k_4+4\sqrt{2}k_2\sqrt{84-240\theta^2+223\theta^4-74\theta^6+8\theta^8})}{14521248-125505072\theta^2+k_5},
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= 1481176\theta-13970088\theta^3+58294432\theta^5-141471600\theta^7+221268845\theta^9-233757712\theta^{11}\\
&\quad +169938520\theta^{13}-84878840\theta^{15}+28501760\theta^{17}-6123520\theta^{19}+757760\theta^{21}-40960\theta^{23},\\
k_2 &= \theta^3\left(12348-66276\theta^2+148543\theta^4-181048\theta^6+130988\theta^8-57584\theta^{10}+15048\theta^{12}-2144\theta^{14}+128\theta^{16}\right),\\
k_3 &= 653420880\theta^4-1833419280\theta^6+3239983790\theta^8-3835461920\theta^{10}+3137639640\theta^{12}-1795305120\theta^{14}\\
&\quad +715888640\theta^{16}-194739200\theta^{18}+34426880\theta^{20}-3563520\theta^{22}+163840\theta^{24},\\
k_4 &= 20744640\theta-157560480\theta^3+535812480\theta^5-1079136000\theta^7+1434589940\theta^9-1327460950\theta^{11}\\
&\quad +877542420\theta^{13}-417840280\theta^{15}+142223920\theta^{17}-33732160\theta^{19}+5287680\theta^{21}-491520\theta^{23}+20480\theta^{25},\\
k_5 &= 4888679040\theta^4-11340070560\theta^6+17443955180\theta^8-18760361370\theta^{10}+14496201740\theta^{12}-8144737400\theta^{14}
\end{aligned}
$$

$$+3328237752\theta^{16}-977815504\theta^{18}+201036680\theta^{20}-27438808\theta^{22}+2232232\theta^{24}-8192\theta^{26}.$$

$\Pi^*_{RM-r1} > \Pi^*_{MM-r1}$ if and only if $\Omega > \hat{\Omega}^{RM-MM}_{r1}(\theta)$ and $\Pi^*_{RM-m1} > \Pi^*_{MM-m1}$ if and only if $\Omega > \hat{\Omega}^{RM-MM}_{m1}(\theta)$. The contour plots clearly show that $\hat{\Omega}^{RM-MM}_{r1}(\theta) > \hat{\Omega}^{RM-MM}_{m1}(\theta)$ and that $\hat{\Omega}^{RM-MM}_{r1}(\theta)$ and $\hat{\Omega}^{RM-MM}_{m1}(\theta)$ increase with θ.

The equilibrium for MR is:

$$\begin{aligned}
D^*_{MR-1} &= \frac{(3-2\theta^2)\left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)}{63-180\theta^2+172\theta^4-64\theta^6+8\theta^8},\\
D^*_{MR-2} &= \frac{(2-\theta^2)\left(\left(21-30\theta^2+8\theta^4\right)A_2-2\theta\left(3-2\theta^2\right)A_1\right)}{2\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)},\\
\Pi^*_{MR-r1} &= \frac{(3-2\theta^2)^2\left(1-\theta^2\right)\left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2},\\
\Pi^*_{MR-r2} &= \frac{(2-\theta^2)^2\left(3-4\theta^2\right)\left(\left(21-30\theta^2+8\theta^4\right)A_2-2\theta\left(3-2\theta^2\right)A_1\right)^2}{16\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2},\\
\Pi^*_{MR-m1} &= \frac{\left(63-144\theta^2+92\theta^4-16\theta^6\right)\left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2}{4\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2},\\
\Pi^*_{MR-m2} &= \frac{(2-\theta^2)\left(6-9\theta^2+2\theta^4\right)\left(\left(21-30\theta^2+8\theta^4\right)A_2-2\theta\left(3-2\theta^2\right)A_1\right)^2}{8\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2}.
\end{aligned}$$

For prices and demands to be nonnegative requires

$$2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\geq 0 \quad \text{and} \quad \left(21-30\theta^2+8\theta^4\right)A_2-2\theta\left(3-2\theta^2\right)A_1\geq 0.$$

The common lower and upper bounds for RM and MM are:

$$\underline{\Omega}^{MR-MM}(\theta)=\frac{\theta\left(2-\theta^2\right)}{2\left(3-4\theta^2+\theta^4\right)} \quad \text{and} \quad \bar{\Omega}^{MR-MM}(\theta)=\frac{21-30\theta^2+8\theta^4}{2\theta\left(3-2\theta^2\right)}.$$

As with RM, the common feasible domain for θ is $\theta\in[0,0.876]$. The boundary lines of $\hat{\Omega}^{MR-MM}_{r2}(\theta)$ and $\hat{\Omega}^{MR-MM}_{m2}(\theta)$ can be obtained by equating the profits of MR and those of MM. Contour plots show that $\hat{\Omega}^{MR-MM}_{r2}(\theta) < \hat{\Omega}^{MR-MM}_{m2}(\theta)$ and that $\hat{\Omega}^{MR-MM}_{r2}(\theta)$ and $\hat{\Omega}^{MR-MM}_{m2}(\theta)$ decrease in θ.

(2) *Compare RR to MR and RM*. The common lower and upper bounds for MR and RR are:

$$\underline{\Omega}^{MR-RR}(\theta)=\frac{\theta\left(3-2\theta^2\right)}{9-14\theta^2+4\theta^4} \quad \text{and} \quad \bar{\Omega}^{MR-RR}(\theta)=\frac{9-14\theta^2+4\theta^4}{\theta\left(3-2\theta^2\right)}.$$

The largest feasible domain for θ is $\theta\in[0,0.823]$, which is the same as that of RR. Boundary values of $\hat{\Omega}^{MR-RR}_{r1}(\theta)$ and $\hat{\Omega}^{MR-RR}_{m1}(\theta)$ result from equating the profits under MR and RR.

$$\hat{\Omega}^{MR-RR}_{r1}(\theta) \;=\; \min\{\hat{\Omega}^{MR-RR}_{r1-1}(\theta),\bar{\Omega}^{MR-RR}(\theta)\},$$

$$\hat{\Omega}_{m1}^{MR-RR}(\theta) \quad = \quad \min\{\hat{\Omega}_{m1-1}^{MR-RR}(\theta), \bar{\Omega}^{MR-RR}(\theta)\},$$

where

$$\begin{aligned}
\hat{\Omega}_{r1-1}^{MR-RR}(\theta) &= \frac{l_1 + 2l_2\sqrt{\left(1-\theta^2\right)^3\left(3-4\theta^2\right)}}{19683 - 244944\theta^2 + l_3}, \\
\hat{\Omega}_{m1-1}^{MR-RR}(\theta) &= \frac{l_4 + \sqrt{2}l_5\sqrt{189 - 642\theta^2 + 700\theta^4 - 264\theta^6 + 32\theta^8}}{91854 - 740664\theta^2 + l_6},
\end{aligned}$$

and where

$$\begin{aligned}
l_1 &= 65661\theta - 583320\theta^3 + 2507776\theta^5 - 6676556\theta^7 \\
&\quad + 11843352\theta^9 - 14362216\theta^{11} + 11980712\theta^{13} - 6822400\theta^{15} + 2589444\theta^{17} - 623336\theta^{19} + 85776\theta^{21} - 512\theta^{23}, \\
l_2 &= \theta^3\left(5103 - 259200\theta^2 + 526320\theta^4 - 551520\theta^6 + 322480\theta^8 - 105920\theta^{10} + 18240\theta^{12} - 1280\theta^{14}\right), \\
l_3 &= 13306668\theta^4 - 41484960\theta^6 + 82566720\theta^8 - 110586720\theta^{10} \\
&\quad + 102338720\theta^{12} - 66050400\theta^{14} + 29577600\theta^{16} - 8989440\theta^{18} + 1766400\theta^{20} - 202240\theta^{22} + 10240\theta^{24}, \\
l_4 &= 306180\theta - 2094660\theta^3 + 6285600\theta^5 - 10910880\theta^7 + 12159840\theta^9 - 9112700\theta^{11} + 4657080\theta^{13} - 1602000\theta^{15} + 354400\theta^{17} - 45440\theta^{19} + 2560\theta^{21}, \\
l_5 &= \theta^3\left(567 - 24390\theta^2 + 41400\theta^4 - 35320\theta^6 + 15920\theta^8 - 3600\theta^{10} + 320\theta^{12}\right), \\
l_6 &= 26435160\theta^4 - 54952560\theta^6 + 73708200\theta^8 - 66821760\theta^{10} + 41719480\theta^{12} - 17934320\theta^{14} + 5207840\theta^{16} - 975040\theta^{18} + 106240\theta^{20} - 5120\theta^{22}.
\end{aligned}$$

$\Pi_{MR-r1}^* > \Pi_{RR-r1}^*$ if and only if $\Omega < \hat{\Omega}_{r1}^{MR-RR}(\theta)$ and $\Pi_{MR-m1}^* > \Pi_{RR-m1}^*$ if and only if $\Omega < \hat{\Omega}_{m1}^{MR-RR}(\theta)$. The contour plots show that $\hat{\Omega}_{r1}^{MR-RR}(\theta) > \hat{\Omega}_{m1}^{MR-RR}(\theta)$ when $\theta \in [0, 0.802]$, where $\hat{\Omega}_{r1}^{MR-RR}(\theta)$ is defined as $\hat{\Omega}_{r1-1}^{MR-RR}(\theta)$, whereas $\hat{\Omega}_{r1}^{MR-RR}(\theta) < \hat{\Omega}_{m1}^{MR-RR}(\theta)$ for $\theta \in [0.802, 0.823]$, where $\hat{\Omega}_{r1}^{MR-RR}(\theta)$ is equivalent to $\hat{\Omega}_{r1-2}^{MR-RR}(\theta)$; $\hat{\Omega}_{r1-1}^{MR-RR}(\theta)$ increases with θ within $[0, 0.630]$, $\hat{\Omega}_{r1-2}^{MR-RR}(\theta)$ decreases with θ within $[0.630, 0.823]$, and $\hat{\Omega}_{m1}^{MR-RR}(\theta)$ increases with θ in the common feasible area.

Boundary lines of $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ and $\hat{\Omega}_{m2}^{RM-RR}(\theta)$ come from equating the profits of RM and RR, where $\hat{\Omega}_{r2}^{RM-RR}(\theta) < \hat{\Omega}_{m2}^{RM-RR}(\theta)$ when $\theta < 0.802$ but the direction of the inequality reverses when $\theta \in [0.802, 0.823]$; $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ decreases with θ within $[0, 0.630]$, where $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ is equivalent to $\hat{\Omega}_{r2-1}^{RM-RR}(\theta)$, and increases with θ within $[0.630, 0.823]$ where $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ is defined as $\hat{\Omega}_{r2-2}^{RM-RR}(\theta)$, and $\hat{\Omega}_{m2}^{RM-RR}(\theta)$ decreases with θ.

(3) *Compare profits between NM (MN) and RM (MR).* The common lower and upper bounds for NM and RM are:

$$\underline{\Omega}^{RM-NM}(\theta) = \frac{2\theta\left(3-2\theta^2\right)}{21 - 30\theta^2 + 8\theta^4} \quad \text{and} \quad \bar{\Omega}^{RM-NM}(\theta) = \frac{2(3-4\theta^2+\theta^4)}{\theta(2-\theta^2)}.$$

The largest feasible domain for θ is $\theta \in [0, 0.876]$. The boundary values of $\hat{\Omega}_{r1}^{RM-NM}(\theta)$ and $\hat{\Omega}_{m1}^{RM-NM}(\theta)$ come from equating the profits in RM and NM.

$$\hat{\Omega}_{r1}^{RM-NM}(\theta) \quad = \quad \min\{\hat{\Omega}_{r1-1}^{RM-NM}(\theta), \bar{\Omega}^{RM-NM}(\theta)\},$$

$$\hat{\Omega}_{m1}^{RM-NM}(\theta) = \min\{\hat{\Omega}_{m1-1}^{RM-NM}(\theta), \bar{\Omega}^{RM-NM}(\theta)\},$$

where

$$\begin{aligned}
\hat{\Omega}_{r1-1}^{RM-NM}(\theta) &= \frac{2(m_1 + 8m_2\sqrt{3 - 7\theta^2 + 4\theta^4})}{4148928 - 357598808\theta^2 + m_3}, \\
\hat{\Omega}_{m1-1}^{RM-NM}(\theta) &= \frac{2(m_4 + 4\sqrt{2}m_2\sqrt{24 - 66\theta^2 + 59\theta^4 - 19\theta^6 + 2\theta})}{8297856 - 708281128\theta^2 + m_5},
\end{aligned}$$

and where

$$\begin{aligned}
m_1 &= 5927040\theta - 45722888\theta^3 + 156707788\theta^5 - 314763000\theta^7 \\
&\quad + 411421650\theta^9 - 367223760\theta^{11} + 228263000\theta^{13} - 98748880\theta^{15} + 29118080\theta^{17} - 5570560\theta^{19} + 622080\theta^{21} - 30720\theta^{23}, \\
m_2 &= \theta^3\left(7056 - 371700\theta^2 + 817870\theta^4 - 979240\theta^6 + 696520\theta^8 - 301280\theta^{10} + 77520\theta^{12} - 10880\theta^{14} + 640\theta^{16}\right), \\
m_3 &= 1378583640\theta^4 - 3137925000\theta^6 + 4689323390\theta^8 - 4840237520\theta^{10} \\
&\quad + 3534734040\theta^{12} - 1839190240\theta^{14} + 676712320\theta^{16} - 171806720\theta^{18} + 28597760\theta^{20} - 2805760\theta^{22} + 122880\theta^{24}, \\
m_4 &= 11854080\theta - 90457920\theta^3 + 308946960\theta^5 - 624137400\theta^7 \\
&\quad + 830360620\theta^9 - 766304790\theta^{11} + 502985060\theta^{13} - 236594360\theta^{15} + 79150960\theta^{17} - 18365760\theta^{19} + 2805760\theta^{21} - 253440\theta^{23} + 10240\theta^{25}, \\
m_5 &= 2723580720\theta^4 - 6236695080\theta^6 + 9473885140\theta^8 - 10068689370\theta^{10} \\
&\quad + 7695641980\theta^{12} - 4281329880\theta^{14} + 1733992720\theta^{16} - 505291520\theta^{18} + 103082240\theta^{20} - 13959680\theta^{22} + 1126400\theta^{24} - 40960\theta^{26}.
\end{aligned}$$

$\Pi^*_{RM-m1} > \Pi^*_{NM-m1}$ if and only if $\Omega > \hat{\Omega}_{m1}^{RM-NM}(\theta)$ and $\Pi^*_{RM-r1} > \Pi^*_{NM-r1}$ if and only if $\Omega > \hat{\Omega}_{r1}^{RM-NM}(\theta)$. The contour plots show that $\hat{\Omega}_{r1}^{RM-NM}(\theta) > \hat{\Omega}_{m1}^{RM-NM}(\theta)$ and that $\hat{\Omega}_{r1}^{RM-NM}(\theta)$ and $\hat{\Omega}_{m1}^{RM-NM}(\theta)$ increase with θ.

The boundary lines of $\hat{\Omega}_{r2}^{MR-MN}(\theta)$ and $\hat{\Omega}_{m2}^{MR-MN}(\theta)$ are obtained by equating the profits in RM and RR, where $\hat{\Omega}_{r2}^{MR-MN}(\theta) < \hat{\Omega}_{m2}^{MR-MN}(\theta)$. $\hat{\Omega}_{r2}^{MR-MN}(\theta)$ and $\hat{\Omega}_{m2}^{MR-MN}(\theta)$ decrease with θ.

(4) *Compare profits between MR (RM) and NR (RN).* The common lower and upper bounds for MR and NR are as follows:

$$\underline{\Omega}^{MR-NR}(\theta) = \frac{\theta\left(2 - \theta^2\right)}{2(3 - 4\theta^2 + \theta^4)} \quad \text{and} \quad \bar{\Omega}^{RM-NM}(\theta) = \frac{4(3 - 4\theta^2 + \theta^4)}{\theta(3 - 2\theta^2)}.$$

The feasible domain for θ is $\theta \in [0, 0.876]$. Define $\Delta\Pi_{m1}^{MR-NR}$ as Manufacturer 1's profit in MR minus the one in NR and $\Delta\Pi_{r1}^{MR-NR}$ as Retailer 1's profit in MR minus the one in NR, which compute to

$$\begin{aligned}
\Delta\Pi_{m1}^{MR-NR} &= -\left(3 - 2\theta^2\right)^2\left(1 - \theta^2\right)\left(\frac{1}{\left(72 - 198\theta^2 + 183\theta^4 - 66\theta^6 + 8\theta^8\right)^2} - \frac{1}{\left(63 - 180\theta^2 + 172\theta^4 - 64\theta^6 + 8\theta^8\right)^2}\right) \\
&\quad \times \left(2\left(3 - 4\theta^2 + \theta^4\right)A_1 - A_2\theta\left(2 - \theta^2\right)\right)^2, \\
\Delta\Pi_{r1}^{MR-NR} &= \frac{1}{4}\left(\frac{63 - 144\theta^2 + 92\theta^4 - 16\theta^6}{\left(63 - 180\theta^2 + 172\theta^4 - 64\theta^6 + 8\theta^8\right)^2} - \frac{4\left(3 - 2\theta^2\right)}{\left(6 - 9\theta^2 + 2\theta^4\right)\left(12 - 15\theta^2 + 4\theta^4\right)^2}\right)
\end{aligned}$$

$$\times \left(2\left(3-4\theta^2+\theta^4\right)A_1-\theta\left(2-\theta^2\right)A_2\right)^2.$$

Graphing shows that

$$\frac{1}{\left(72-198\theta^2+183\theta^4-66\theta^6+8\theta^8\right)^2}-\frac{1}{\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2}<0$$

and

$$\frac{63-144\theta^2+92\theta^4-16\theta^6}{\left(63-180\theta^2+172\theta^4-64\theta^6+8\theta^8\right)^2}-\frac{4\left(3-2\theta^2\right)}{\left(6-9\theta^2+2\theta^4\right)\left(12-15\theta^2+4\theta^4\right)^2}>0$$

for any $\theta \in [0, 0.876]$. Thus $\Delta\Pi_{m1}^{MR-NR} > 0$ and $\Delta\Pi_{r1}^{MR-NR} > 0$ in the common feasible area, meaning that Manufacturer 1 and Retailer 1 always prefer MR to NR. Similarly, $\Delta\Pi_{r2}^{RM-RN} > 0$ and $\Delta\Pi_{m2}^{RM-RN} > 0$ for any $\theta \in [0, 0.876]$. This completes the proof of Lemma 6.

Lemma 6 suggests that both Manufacturer 1 and Retailer 1 would have incentives to switch from MM to RM, if their supply chain has a larger base demand than the other, and these incentives become stronger with higher product substitutability. This occurs because retailer advertising intensifies competition relative to manufacturer advertising (i.e., the levels of retailer advertising are higher in equilibrium, whose impact plays out through the demand function in Eq. (1)). However, an area exists (i.e., $\hat{\Omega}_{m1}^{RM-MM}(\theta) < \Omega < \hat{\Omega}_{r1}^{RM-MM}(\theta)$) within which Manufacturer 1 and Retailer 1 cannot agree on whether to use MM or RM. A similar situation arises with regards to RM and NM. Between RR and MR, Manufacturer 1 and Retailer 1 both prefer MR to RR if the supply chain's base market is the smaller one, but reverse their preference if the base market is the larger, especially when product substitutability is high. Manufacturer 1 and Retailer 1 always prefer MR to NR, because the manufacturer advertising yields significantly more demand for the supply chain but without greatly intensifying the supply chain competition. Similar sentiments govern the preferences of Manufacturer 2 and Retailer 2 as they consider switching from MM to MR, RR to RM, MR to MN, and RM to RN. To summarize, both manufacturer and both retailers prefer retailer advertising when product substitutability is low; when product substitutability is high, manufacturer advertising has some appeal. Lemma 6 also indicates that RN/NR are inferior to RM/MR. These findings, along with Theorems 1 and 2, indicate that MM, RR, RM, and MR are more stable than the other advertising structures.

A state is a strong channel equilibrium if no coalition of players in the same supply chain can profitably deviate from the current state. It can be shown that other advertising structures, including hybrid approaches MN, NM, RN, and NR, are dominated by MM, RR, RM, and MR. So the following will focus on evaluating MM, RR, RM, and MR for the manufacturers and retailers. We continue to consider only the common feasible domain established in Lemma 6.

We start with MM. The proof of Lemma 6 established that Manufacturer 1 prefers RM to MM if and only if $\Omega > \hat{\Omega}_{m1}^{RM-MM}(\theta)$ and Retailer 1 prefers RM to MM if and only if $\Omega > \hat{\Omega}_{r1}^{RM-MM}(\theta)$. Given that $\hat{\Omega}_{r1}^{RM-MM}(\theta) > \hat{\Omega}_{m1}^{RM-MM}(\theta)$, the coalition of Manufacturer 1 and Retailer 1 would never switch from MM to RM as long as $\Omega < \max\{\hat{\Omega}_{r1}^{RM-MM}(\theta), \hat{\Omega}_{m1}^{RM-MM}(\theta)\} = \hat{\Omega}_{r1}^{RM-MM}(\theta)$, because at least one of Manufacturer 1 and Retailer 1 will be worse off switching from MM to RM. On the other hand, for Manufacturer 2, MR outperforms MM if and only if $\Omega < \hat{\Omega}_{m2}^{MR-MM}(\theta)$; whereas for Retailer 2, MR outperforms MM if and only if $\Omega < \hat{\Omega}_{r2}^{MR-MM}(\theta)$. Given that $\hat{\Omega}_{r2}^{MR-MM}(\theta) < \hat{\Omega}_{m2}^{MR-MM}(\theta)$, similarly, the coalition of Manufacturer 1 and Retailer 1 would never switch from MM to MR as long as

$$\Omega > \min\{\hat{\Omega}_{r2}^{MR-MM}(\theta), \hat{\Omega}_{m2}^{MR-MM}(\theta)\} = \hat{\Omega}_{r2}^{MR-MM}(\theta).$$

Therefore, MM is a strong channel equilibrium if $\hat{\Omega}_{r2}^{MR-MM}(\theta) < \Omega < \hat{\Omega}_{r1}^{RM-MM}(\theta)$.

Consider RR. Manufacturer 1 prefers MR to RR if and only if $\Omega < \hat{\Omega}_{m1}^{MR-RR}(\theta)$ and Retailer 1 prefers MR to RR if and only if $\Omega < \hat{\Omega}_{r1}^{MR-RR}(\theta)$. Given that $\hat{\Omega}_{r1}^{MR-RR}(\theta)$ and $\hat{\Omega}_{m1}^{MR-RR}(\theta)$ cross in the common feasible domain, it is conceivable that the coalition of Manufacturer 1 and Retailer 1 would never switch from RR to MR as long as $\Omega > \min\{\hat{\Omega}_{r1}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\}$. On the other hand, Manufacturer 2 prefers RM to RR if and only if $\Omega > \hat{\Omega}_{m2}^{RM-RR}(\theta)$ and Retailer 2 prefers RM to RR if and only if $\Omega > \hat{\Omega}_{r2}^{RM-RR}(\theta)$. Given that $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ crosses $\hat{\Omega}_{m2}^{RM-RR}(\theta)$ at $\theta = 0.802$, it is conceivable that no coalition of both Manufacturer 2 and Retailer 2 would switch from RR to RM as long as $\Omega < \max\{\hat{\Omega}_{r2}^{RM-MM}(\theta), \hat{\Omega}_{m2}^{RM-MM}(\theta)\}$. So RR is a strong channel equilibrium if and only if

$$\min\{\hat{\Omega}_{r1}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\} < \Omega < \max\{\hat{\Omega}_{r2}^{RM-MM}(\theta), \hat{\Omega}_{m2}^{RM-MM}(\theta)\}.$$

And since $\hat{\Omega}_{m2}^{RM-RR}(\theta)$ bypasses $\hat{\Omega}_{m1}^{MR-RR}(\theta)$ at $\theta = 0.775$ before reaching $\hat{\Omega}_{r1}^{MR-RR}(\theta)$ and $\hat{\Omega}_{r2}^{RM-MM}(\theta)$, we conclude that RR is a strong channel equilibrium if $\hat{\Omega}_{m1}^{MR-RR}(\theta) < \Omega < \hat{\Omega}_{m2}^{RM-RR}(\theta)$ in $\theta \in [0, 0.775]$.

Consider MR. Manufacturer 1 prefers RR to MR if and only if $\Omega > \hat{\Omega}_{m1}^{MR-RR}(\theta)$ and Retailer 1 prefers RR to MR if and only if $\Omega > \hat{\Omega}_{r1}^{MR-RR}(\theta)$. Given that $\hat{\Omega}_{r1}^{MR-RR}(\theta)$ and $\hat{\Omega}_{m1}^{MR-RR}(\theta)$ cross in the common feasible domain, it is conceivable that the coalition of both Manufacturer 1 and Retailer 1 would never switch from MR to RR as long as $\Omega < \max\{\hat{\Omega}_{r1}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\}$. On the other hand, for Manufacturer 2 MM outperforms MR if and only if $\Omega > \hat{\Omega}_{m2}^{MR-MM}(\theta)$, whereas for Retailer 2, MM outperforms MR if and only if $\Omega > \hat{\Omega}_{r2}^{MR-MM}(\theta)$. Given that $\hat{\Omega}_{r2}^{MR-MM}(\theta) < \hat{\Omega}_{m2}^{MR-MM}(\theta)$, the coalition of Manufacturer 1 and Retailer 1 would never switch from MR to MM as long as $\Omega < \max\{\hat{\Omega}_{r2}^{MR-MM}(\theta), \hat{\Omega}_{m2}^{MR-MM}(\theta)\} = \hat{\Omega}_{m2}^{MR-MM}(\theta)$. Therefore, MR is a strong channel equilibrium as long as $\Omega < \min\{\hat{\Omega}_{m2}^{MR-MM}(\theta), \max\{\hat{\Omega}_{r1}^{MR-RR}(\theta), \hat{\Omega}_{m1}^{MR-RR}(\theta)\}\}$.

Consider RM. Manufacturer 1 prefers MM to RM if and only if $\Omega < \hat{\Omega}_{m1}^{RM-MM}(\theta)$ and Retailer 1 prefers MM to RM if and only if $\Omega < \hat{\Omega}_{r1}^{RM-MM}(\theta)$. Given that $\hat{\Omega}_{r1}^{RM-MM}(\theta) > \hat{\Omega}_{m1}^{RM-MM}(\theta)$, it is conceivable that the coalition of Manufacturer 1 and Retailer 1 would never switch from MM to RM as long as

$$\Omega > \min\{\hat{\Omega}_{r1}^{RM-MM}(\theta), \hat{\Omega}_{m1}^{RM-MM}(\theta)\} = \hat{\Omega}_{m1}^{RM-MM}(\theta).$$

On the other hand, Manufacturer 2 prefers RR to RM if and only if $\Omega < \hat{\Omega}_{m2}^{RM-RR}(\theta)$ and Retailer 2 prefers RR to RM if and only if $\Omega < \hat{\Omega}_{r2}^{RM-RR}(\theta)$. Given that $\hat{\Omega}_{r2}^{RM-RR}(\theta)$ crosses $\hat{\Omega}_{m2}^{RM-RR}(\theta)$ at $\theta = 0.802$, it is conceivable that the coalition of Manufacturer 2 and Retailer 2 would never switch from RM to RR as long as

$$\Omega > \min\{\hat{\Omega}_{r2}^{RM-MM}(\theta), \hat{\Omega}_{m2}^{RM-MM}(\theta)\}.$$

Therefore, RM is a strong channel equilibrium as long as

$$\Omega > \max\{\hat{\Omega}_{m1}^{RM-MM}(\theta), \min\{\hat{\Omega}_{r2}^{RM-MM}(\theta), \hat{\Omega}_{m2}^{RM-MM}(\theta)\}\}.\square$$

Proof of Lemma 5: Solving the Nash game gives

$$e_{m1} = \frac{(3-2\theta^2)\,(15-2\theta\,(3+\theta\,(13-2\theta-4\theta^2)))}{45(-1+6k_{m1})+12(9-77k_{m1})\theta^2+4(-19+260k_{m1})\theta^4+16(1-28k_{m1})\theta^6+64k_{m1}\theta^8}.$$

Differentiating this yields

$$\frac{\partial e_{m1}}{\partial k_{m1}} = -\frac{(3-2\theta^2)\,(270-924\theta^2+1040\theta^4-448\theta^6+64\theta^8)\,(15-2\theta\,(3+\theta\,(13-2\theta-4\theta^2)))}{(45(-1+6k_{m1})+12(9-77k_{m1})\theta^2+4(-19+260b)\theta^4+16(1-28k_{m1})\theta^6+64k_{m1}\theta^8)^2},$$

which is nonpositive if and only if $15 - 2\theta\left(3 + \theta\left(13 - 2\theta - 4\theta^2\right)\right) \geq 0$, which is true under the assumptions that keep demand nonnegative. Therefore, Manufacturer 1's advertising effort decreases with k_{m1}. $\square$