Should Central Banks Worry about Nonlinearities of Their Large-Scale Macroeconomic Models?

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Large-scale models for policy analysis

Mostly used by central banks and government agencies:

- International Monetary Fund's Global Economy Model, GEM (Bayoumi et al., 2001);
- US Federal Reserve Board's SIGMA model (Erceg et al., 2006);
- Bank of Canada Terms of Trade Economic Model, ToTEM (Dorich et al. 2013);
- European Central Bank's New Area-Wide Model, NAWM (Coenen et al. 2008);
- Bank of England COMPASS model (Burgess et al., 2013);
- Swedish Riksbank's Ramses II model (Adolfson et al., 2013).

Requirements to central bank models

1. Central bank models must mimic as close as possible the actual economies in every possible dimension.
   - *Then, the policymakers can produce realistic simulation of alternative policies and to choose the best one.*
2. Central bank models must be rich and flexible enough to describe interactions between many variables of interest, including different types of foreign and domestic consumption, investment, capital, labor, prices, exchange rate, etc.
   - *Central bank models may contain hundreds of equations.*
   - *Their estimation, calibration, solution and simulation are highly nontrivial tasks.*
3. Central bank models need DSGE models for policy analysis.
   - *Econometric models have limitations for policy analysis (Lucas critique).*

Numerical methods used by central banks

- The central banks use linear (first-order) perturbation methods – Linear Taylor's expansions.
- *Advantages:*
  – computationally inexpensive;
  – simple to use;
  – can be applied to very large problems.
- *Drawbacks*:
  – insufficiently accurate in the presence of strong nonlinearities;
  – neglect second-order effects of the volatility of shocks on numerical solutions.
- Nonlinear effects can be economically significant; see Judd et al. (2017).

Questions addressed in the paper

1. How large could be the difference between local linear and global nonlinear solutions in realistically calibrated central banking models?

2. Could the limitations of the first-order perturbation analysis distort policy implications of realistic central banking models?

The answers to these questions are unknown as no one has computed nonlinear solutions to large-scale central banks' models.

Project of Bank of Canada

To answer these questions, the Bank of Canada created a working group whose objective is to construct global solutions to their large-scale models.

The results of this project are summarized in the form of a technical report and a research paper.

In my presentation, I outline the results produced by this working group.
Full scale ToTEM model of Bank of Canada

- The Terms of Trade Economic Model (ToTEM) – the main projection and policy analysis model of the Bank of Canada.
- Small-open economy model.
- ToTEM includes 356 equations and unknowns
$\Longrightarrow$ It is too large for the existing global solution methods.

A scaled-down version of ToTEM

- We construct a scaled-down version of ToTEM, which we call a "baby ToTEM" (bToTEM) model.
- bToTEM includes 49 equations and unknowns
  $\Longrightarrow$ It is still a large-scale model.
- Two production sectors: final-good production and commodity production.
- Meaningful trade: final goods, commodities, imports.
- One representative household, with differentiated labour services.
- Taylor-type interest rate rule
- Six shocks, exogenous ROW.

Differences between ToTEM and bToTEM

- *ToTEM*
  – 5 distinct production sectors (consumption goods and services, investment goods, government goods, noncommodity export goods, and commodities);
  – 4 sectors are identical except of parameters, while the commodity sector is different;
  – A separate economic model of the rest of the world (ROW).
- *bToTEM*
  – The final-good production sector is identical in structure to the consumption goods and services sector of ToTEM;
  – Linear technologies for transforming the output of this sector into other types of output that correspond to the remaining ToTEM's sectors;
  – The ROW sector is modeled using exogenous processes for foreign variables.
- Three Phillips curves in bToTEM but eight of them in ToTEM.

Production of final goods

Two stages:

1. In the first stage, intermediate goods are produced by identical perfectly competitive firms from labour, capital, commodities, and imports.
2. In the second stage, a variety of final goods are produced by monopolistically competitive firms from the intermediate goods. The variety of final goods is then aggregated into the final consumption good.

First stage of production

- Perfectly competitive firms produce an intermediate good:

$$Z_t^g = \left(\delta_l \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_k \left(u_t K_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_{com} \left(COM_t^d\right)^{\frac{\sigma-1}{\sigma}} + \delta_m \left(M_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \quad (1)$$

L_t, K_t, COM_t^d = labour, capital and commodity inputs, resp., M_t = imports, u_t = capital utilization, A_t = the level of labour-augmenting technology,

$$\log\left(A_t\right) = \varphi_a \log\left(A_{t-1}\right) + \left(1-\varphi_a\right)\log\left(\bar{A}\right) + \xi_t^a. \quad (2)$$

- Capital depreciates according to the following law of motion

$$K_{t+1} = \left(1-d_t\right) K_t + I_t, \quad (3)$$

where d_t is the depreciation rate, and I_t is investment.

- The depreciation rate increases with capital utilization:

$$d_t = d_0 + \overline{d} e^{\rho\left(u_t-1\right)}. \quad (4)$$

First stage of production (cont.)

- The firms incur a quadratic adjustment cost when adjusting the level of investment. The net output is given by

$$Z_t^n = Z_t^g - \frac{\chi_i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 I_t. \tag{5}$$

- The objective of the firms is to choose L_t, K_{t+1}, I_t, COM_t, M_t, u_t in order to maximize profits

$$E_0 \sum_{t=0}^{\infty} \mathcal{R}_{0,t}\left(P_t^z Z_t^n - W_t L_t - P_t^{com} COM_t^d - P_t^i I_t - P_t^m M_t\right)$$

s.t. $(1) - (5)$,

$\mathcal{R}_{t,t+j} = \beta^j \left(\lambda_{t+j}/\lambda_t\right)\left(P_t/P_{t+j}\right) =$ stochastic discount factor.

Second stage of production

- A monopolistically competitive firm $i \in [0,1]$ produces a differentiated good

$$Z_{it} = \min\left(\frac{Z_{it}^{n}}{1-s_m}, \frac{Z_{it}^{mi}}{s_m}\right),$$

Z_{it}^{n} = intermediate good; Z_{it}^{mi} = manufactured input;
s_m = a Leontief parameter.

- The differentiated goods Z_{it} are aggregated into the final good Z_t according to a CES function:

$$Z_t = \left(\int_0^1 Z_{it}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

- Cost minimization implies

$$Z_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Z_t,$$

where $P_t = \left(\int_0^1 P_{it}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$.

Second stage of production (cont.)

- The final good is used as the manufactured inputs by each of the monopolistically competitive firms.
- Two types of monopolistically competitive firms: rule-of-thumb firms of measure ω and forward-looking firms of measure $1-\omega$.
- Within each type, with probability θ the firms index their price to the (time-varying) inflation target $\bar{\pi}_t$.
- The rule-of-thumb firms, which do not index their price in the current period, partially index their price:

$$P_{it} = (\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma} P_{i,t-1}.$$

Second stage of production (cont.)

- The optimizing forward-looking firms solve:

$$\max_{P_t^*} E_t \left\{ \sum_{j=0}^{\infty} \theta^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^* Z_{i,t+j} \right.\right. \\ \left.\left. - (1 - s_m) P_{t+j}^z Z_{i,t+j} - s_m P_{t+j} Z_{i,t+j} \right) \right\}$$

subject to demand constraints

$$\text{s.t.} Z_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^*}{P_{t+j}} \right)^{-\varepsilon} Z_{t+j}.$$

Imports

- Intermediate imported goods M_{it} are bounded into the final imported good M_t according to

$$M_t = \left(\int_0^1 M_{it}^{\frac{\varepsilon_m - 1}{\varepsilon_m}} di \right)^{\frac{\varepsilon_m}{\varepsilon_m - 1}}.$$

- The demand for an intermediate imported good i:

$$M_{it} = \left(\frac{P_{it}^m}{P_t^m} \right)^{-\varepsilon_m} M_t,$$

where $P_t^m = \left(\int_0^1 \left(P_{it}^m \right)^{1-\varepsilon_m} di \right)^{\frac{1}{1-\varepsilon_m}}$.

- The price of intermediate imported goods is set in the currency of the importing country.

Import (cont.)

- The optimizing forward-looking firms solves

$$\max_{P_t^{m*}} E_t \left[\sum_{j=0}^{\infty} (\theta_m)^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*} M_{i,t+j} - e_{t+j} P_{t+j}^{mf} M_{i,t+j} \right) \right]$$

subject to demand constraints

$$M_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*}}{P_{t+j}^{m}} \right)^{-\varepsilon_m} M_{t+j},$$

P_t^{mf} = foreign price of imports; e_t = nominal exchange rate (domestic price of a unit of foreign currency).

Foreign demand and foreign economy

- The foreign demand for Canadian noncommodity exports is given by the demand function

$$X_t^{nc} = \gamma^f \left(\frac{P_t^{nc}}{e_t P_t^f} \right)^{-\varphi} Z_t^f,$$

$P_t^{nc}/e_t =$ foreign price of noncommodity exports;
$P_t^f =$ foreign general price level.

- The balance of payments

$$\frac{e_t B_t^f}{R_t^f \left(1 + \kappa_t^f\right)} - e_t B_{t-1}^f = P_t^{nc} X_t^{nc} + P_t^{com} X_t^{com} - P_t^m M_t,$$

$B_t^f =$ domestic holdings of foreign-currency denominated bonds.;
$\kappa_t^f =$ risk premium.

Foreign demand and foreign economy (cont.)

- The rest of the world is specified by three exogenous processes that describe the evolution of foreign variables.
- The foreign demand for Canadian noncommodity exports Z_t^f

$$\log\left(Z_t^f\right) = \varphi_{Zf}\log\left(Z_{t-1}^f\right) + \left(1-\varphi_{zf}\right)\log\left(\bar{Z}^f\right) + \xi_t^{zf}.$$

- A foreign interest rate shock r_t^f

$$\log\left(r_t^f\right) = \varphi_{rf}\log\left(r_{t-1}^f\right) + \left(1-\varphi_{rf}\right)\log\left(\bar{r}\right) + \xi_t^{rf}.$$

- A foreign commodity price p_t^{comf} is

$$\log\left(p_t^{comf}\right) = \varphi_{comf}\log\left(p_{t-1}^{comf}\right) + \left(1-\varphi_{comf}\right)\log\left(\bar{p}^{comf}\right) + \xi_t^{comf},$$

ξ_t^{zf}, ξ_t^{rf}, ξ_t^{comf} = normally distributed random variables;
φ_{Zf}, φ_{rf}, φ_{comf} = autocorrelation coefficients.

Commodity production

- The commodities are produced from the final goods by representative, perfectly competitive domestic firms:

$$COM_t = (Z_t^{com})^{s_z} (A_t F)^{1-s_z} - \frac{\chi_{com}}{2}\left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1\right)^2 Z_t^{com},$$

F = a fixed production factor (land).
Here, the second term is quadratic adjustment costs.

- The commodities are sold at the rest-of-the-world price adjusted by the nominal exchange rate

$$P_t^{com} = e_t P_t^{com\,f}.$$

- The commodities are sold domestically (COM_t^d) or exported to the rest of the world (X_t^{com})

$$COM_t = COM_t^d + X_t^{com}.$$

Households

- The representative household's period utility function:

$$U_t = \frac{\mu}{\mu - 1} \left(C_t - \xi \bar{C}_{t-1} \right)^{\frac{\mu - 1}{\mu}} \exp\left(\frac{\eta (1 - \mu)}{\mu (1 + \eta)} \int_0^1 (L_{ht})^{\frac{\eta + 1}{\eta}} \, dh \right) \eta_t^c ,$$

C_t = consumption of finished goods;
$\bar{C}_t$ = aggregate consumption;
L_{ht} = labour service of type h;
η_t^c = a consumption demand shock.

- We assume

$$\log \left(\eta_t^c \right) = \varphi_c \log \left(\eta_{t-1}^c \right) + \xi_t^c ,$$

ξ_t^c = a normally distributed variable; φ_c = an autocorrelation coefficient.

Households (cont.)

- The representative household solves

$$E_t\left[\sum_{j=0}^{\infty}\beta^t U_{t+j}\right]$$

s.t. $P_tC_t+\frac{B_t}{R_t}+\frac{e_tB_t^f}{R_t^f\left(1+\kappa_t^f\right)}=B_{t-1}+e_tB_{t-1}^f+\int_0^1 W_{ht}L_{ht}dh+\Pi_t,$

B_t = holdings of domestic bonds;
B_t^f = holdings of foreign-currency denominated bonds;
Π_t are profits paid by the firms.

- To induce the stationarity of the model, we assume that the risk premium κ_t^f is

$$\kappa_t^f=\varsigma\left(\bar{b}^f-b_t^f\right),$$

$b_t^f=e_tB_t^f/\left(\pi_{t+1}^f P_t\bar{Y}\right)$ = normalized bond holdings; see Schmitt-Grohé and Uribe (2003).

Wage setting

- Households;
- Aggregating firms (labor packers) – in perfect competition;
- Labour unions – in monopolistic competition.

Labour packers

- The representative household supplies a variety of differentiated labour service L_{ht}, $h \in [0,1]$ to the labour market.
- The differentiated labour service is aggregated by labor packers according to

$$L_t = \left(\int_0^1 L_{ht}^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}.$$

- L_t is used in the first stage of production.
- Cost minimization of the labor packer implies the demand

$$L_{ht} = \left(\frac{W_{ht}}{W_t} \right)^{-\varepsilon_w} L_t,$$

$W_{ht} =$ wage for labour of type h; $W_t \equiv \left(\int_0^1 W_{ht}^{1-\varepsilon_w} dh \right)^{\frac{1}{1-\varepsilon_w}}$.

Labour unions (cont.)

- Two types: rule-of-thumb (measure ω_w) and forward-looking (measure $1-\omega_w$) unions.
- Within each type, with probability θ_w a union indexes wage to the inflation target $\bar{\pi}_t$, $W_{it} = \bar{\pi} W_{i,t-1}$.
- The rule-of-thumb unions which do not index their wage set

$$W_{it} = \left(\pi_{t-1}^{w}\right)^{\gamma_w} \left(\bar{\pi}_t\right)^{1-\gamma_w} W_{i,t-1}.$$

- The forward-looking unions that do not index their wage solve:

$$\max_{W_t^*} E_t \left[\sum_{j=0}^{\infty} \left(\beta\theta_w\right)^j U_{t+j} \right]$$

$$\text{s.t. } L_{h,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^*}{W_{t+j}} \right)^{-\varepsilon_w} L_{t+j},$$

$$P_{t+j} C_{t+j} = \prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^* L_{h,t+j} dh + \dots$$

Monetary policy

- Taylor rule:

$$\begin{aligned} R_t &= \rho_r R_{t-1} + (1-\rho_r)\left[\bar{R} + \rho_\pi\left(\pi_t - \bar{\pi}_t\right) + \rho_Y\left(\log Y_t - \log \bar{Y}_t\right)\right] + \eta_t^r \\ &\equiv \Phi_t, \end{aligned}$$

  $\bar{Y}_t$ = potential output; η_t^r = interest rate shock,

$$\eta_t^r = \varphi_r \eta_{t-1}^r + \xi_t^r,$$

  ξ_t^r = a normally distributed variable; φ_r = autocorrelation coefficient.

- Potential output changes with productivity as

$$\log \bar{Y}_t = \varphi_{zf} \log \bar{Y}_{t-1} + \left(1-\varphi_{zf}\right) \log\left(\frac{A_t \bar{Y}}{\bar{A}}\right).$$

- If an effective lower bound (ELB) is imposed on the nominal interest rate, R_t^{ELB}, then

$$R_t = \max\left\{\Phi_t, R_t^{ELB}\right\}.$$

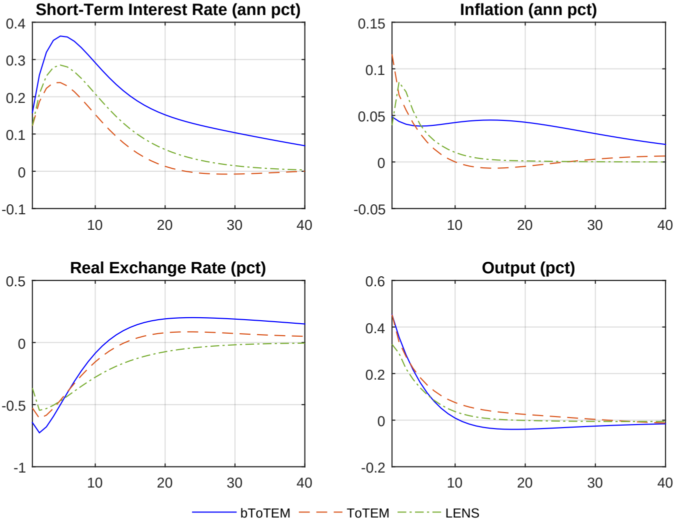
Calibration

- 61 parameters to calibrate.
- Whenever possible, we use the same parameters as in ToTEM.
- We choose the remaining parameters to reproduce observations on the Canadian economy.
- We target the ratios of the following variables to nominal GDP:
    - consumption,
    - investment,
    - noncommodity export,
    - commodity export,
    - import,
    - total commodities,
    - labor input.

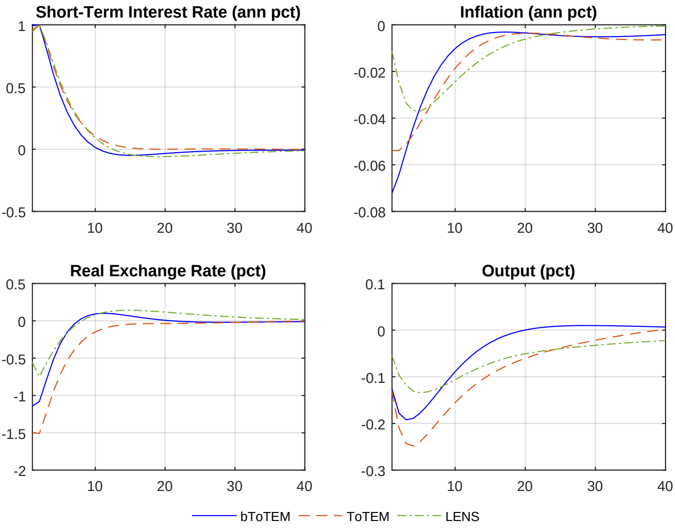
A comparison of bToTEM to ToTEM and LENS

- The Bank of Canada uses a first-order perturbation method to solve ToTEM.
- For ToTEM, we use IRIS – open-source software used by the Bank of Canada for macroeconomic modeling.
- For bToTEM, we use IRIS Toolbox, as well as Dynare.
- We checked that the IRIS and Dynare packages produce indistinguishable numerical solutions for bToTEM.
- Also, we include for comparison the LENS model – another model of the Bank of Canada.
- LENS is a semistructural model.
- Both, the ToTEM and LENS models, include more shocks than bToTEM:
    - 52 shocks in ToTEM and 98 shocks in LENS.

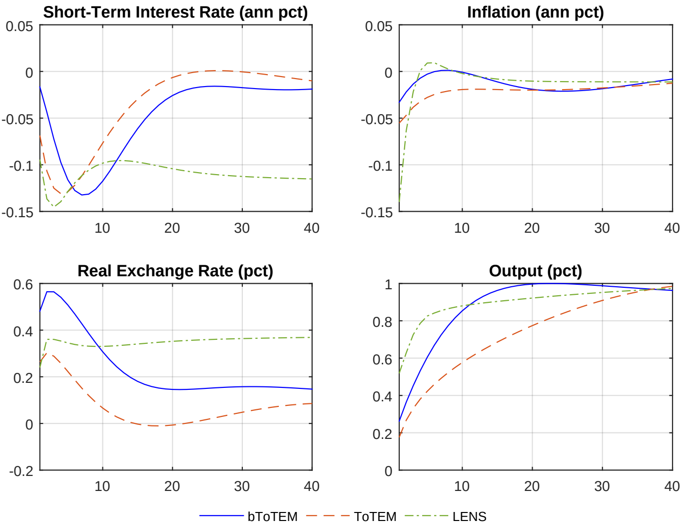
Impulse response to a consumption demand shock



Impulse response to an interest rate shock



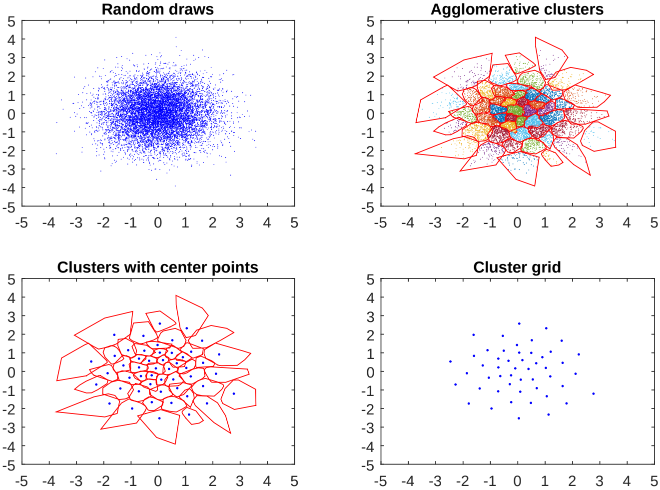
Impulse response to a permanent productivity shock



bToTEM: a serious challenge for global methods

- The models like bToTEM has not been yet studied in the literature using global methods.
- bToTEM contains 21 state variables (6 exogenous and 15 endogenous ones) $\Longrightarrow$ *curse of dimensionality*.
- Moreover, the bToTEM's equations are complex and require the use of numerical solvers.
- The state-of-the-art **cluster-grid algorithm (CGA)** of Maliar and Maliar (2015).
- CGA is a projection-style global solution method that uses adaptive grid
  - the model is solved only in the area of the state space visited in simulation
  - relies on integration and optimization methods are tractable in high-dimensional problems
- CGA can accurately solve models with dozens of state variables.

Example: Construction of a cluster grid



Understanding the role of nonlinearities in the solution

Potential effects of nonlinearities on the properties of the solution compared to a plain linearization method:

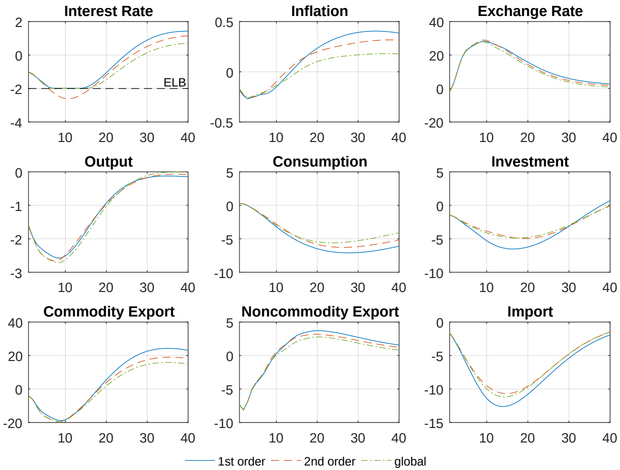
- **(ELB)** The ELB kink in the Taylor rule can induce kinks and nonlinearities in other variables of the model
- **(Higher order terms)** Higher order terms, neglected by linearization, can be quantitatively important for the properties of the solution
- **(Solution domain)** The quality of local (perturbation) solutions, constructed to be accurate in the steady state, can deteriorate when deviating from the steady state.

Experiment 1: Foreign-driven recession

- The U.S. is the main Canadian trade partner (around 75% of Canadian exports goes to the US).
- In 2008, the Canadian economy experienced a huge 16% drop in exports.
- In 2009–2010, the Bank of Canada targeted the overnight interest rate at 0.25% annually (the lower bound).
- To model the ROW:
  - we produce impulse responses for 3 ROW variables (interest rate, commodity price, output) from ToTEM.
  - we use them as exogenous shocks in the bToTEM model.
- ROW output in ToTEM declines by 7% on the impact of shock and by 12% at the peak – the numbers consistent with the magnitudes during the Great Recession.

Experiment 1: Foreign-driven recession

Linear local, quadratic local and global solutions



Approximation errors

Residuals in the model's equations on the simulated path, $log10$ units

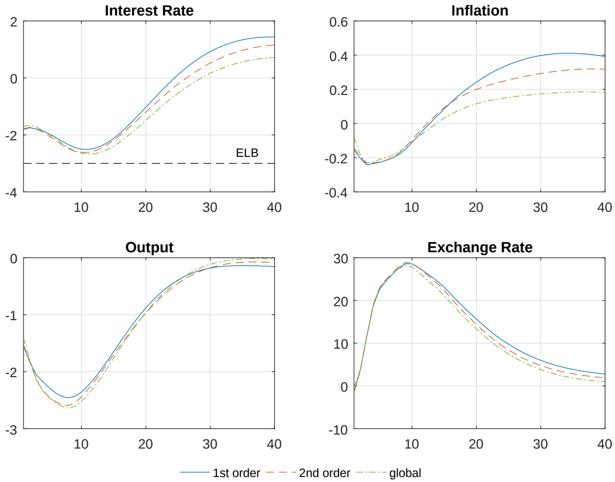
	Maximum error			Average error		
	Local 1st order	Local 2nd order	Global 2nd order	Local 1st order	Local 2nd order	Global 2nd order
R_t	-3.83	-3.84	-5.07	-4.13	-4.53	-5.74
π_t	-4.40	-3.81	-4.38	-4.65	-4.49	-5.08
s_t	-2.38	-1.96	-2.51	-3.16	-2.62	-3.24
Y_t	-2.59	-3.27	-3.42	-2.78	-3.90	-4.16
C_t	-3.19	-3.13	-3.94	-3.44	-3.79	-4.74
I_t	-3.01	-3.39	-3.51	-3.79	-4.01	-4.09
X_t^{com}	-1.75	-2.32	-2.73	-1.97	-2.88	-3.43
X_t^{nc}	-2.78	-2.36	-2.91	-3.56	-3.01	-3.64
M_t	-2.17	-2.90	-3.17	-2.41	-3.53	-3.88
Average	-2.76	-2.81	-3.41	-3.20	-3.45	-4.11
Max	-1.43	-1.44	-2.09	-1.92	-2.08	-2.75

Experiment 2: Higher inflation target

- For the last 25 years, the Bank of Canada adhered to the inflation targeting, however, every three to five years it revises their inflation-control framework.
- Current inflation target is 2%.
- A higher inflation target could be beneficial by reducing frequency and severity of ELB episodes.
- Kryvtsov and Mendes (2015), Dorich et al. (2017).
- We use the bToTEM model to assess the effects of an increase in the inflation target from 2% to 3%.

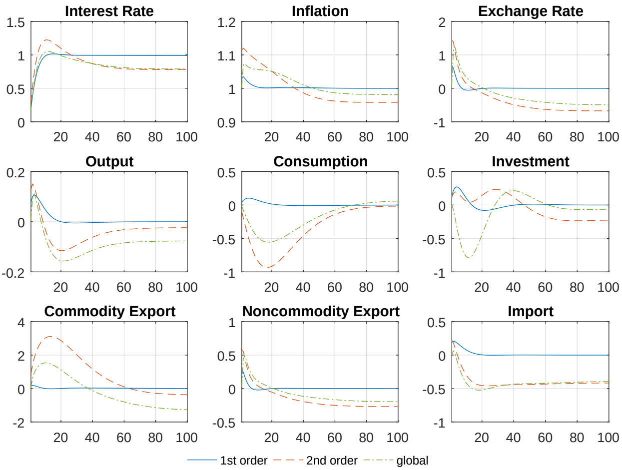
Experiment 2: Higher inflation target

A 3% inflation target could prevent ELB episodes similar to the 2009-10 episode



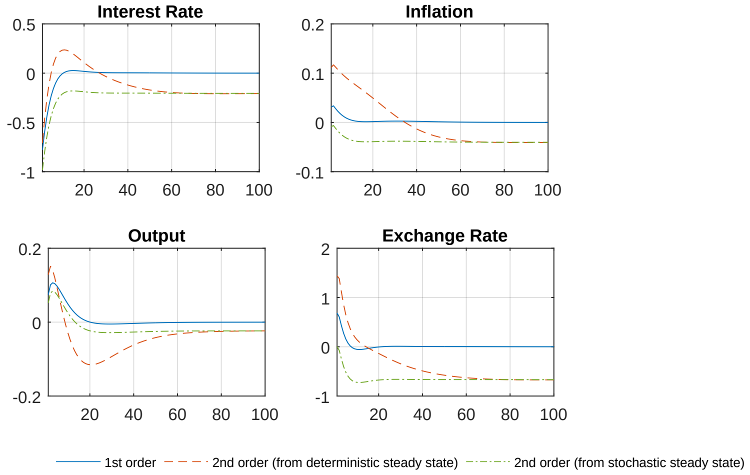
Experiment 2: Higher inflation target

Transition from the deterministic steady state with the inflation target of 2%



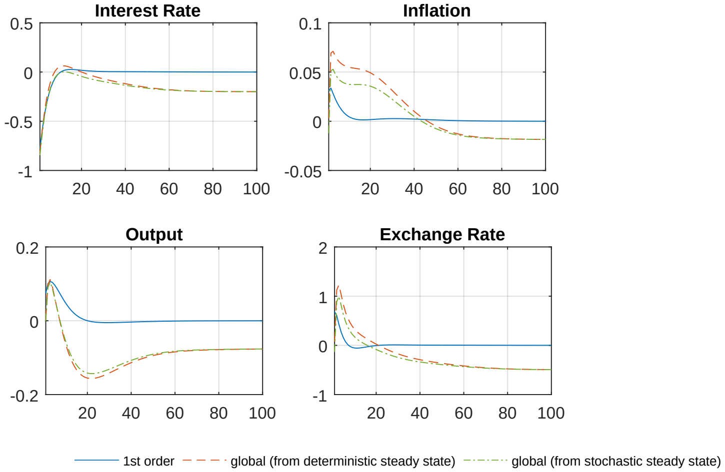
Experiment 2: Higher inflation target

Transition from the different states with the target of 2%, 2nd order solution



Experiment 2: Higher inflation target

Transition from the different states with the target of 2%, global solution



Closing condition

Revisiting the stationarity condition of Schmitt-Grohé and Uribe (2003):

- closing condition in an exponential form, used in Schmitt-Grohé and Uribe (2003)

$$\kappa_t^f = \varsigma \left[\exp\left(\bar{b}^f - b_t^f\right) - 1 \right]$$

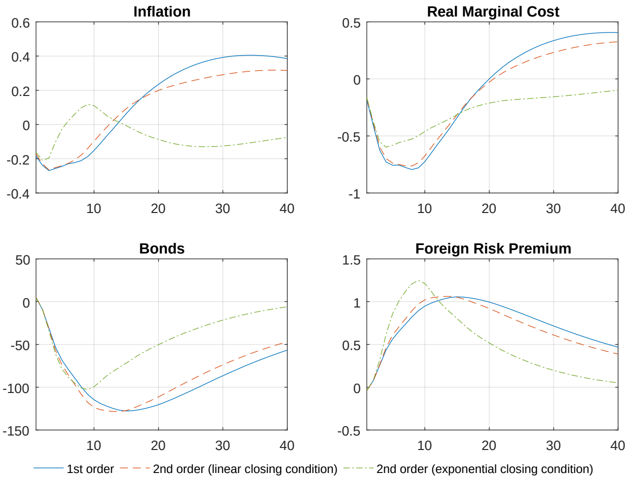
- linear closing condition

$$\kappa_t^f = \varsigma \left(\bar{b}^f - b_t^f\right)$$

- the two conditions are equivalent up to the first order

Nonlinearities are important, finally!

Transition under different closing conditions



Conclusion

- bToTEM model
  - can complement the existing models of the Bank of Canada
  - tractable with global nonlinear solution methods
  - accuracy can be assessed
- Linear vs. nonlinear solutions
  - the role of nonlinearities is modest in a realistically calibrated bToTEM model
  - apparently innocent changes in the model's assumptions can make nonlinearity important
- Suggestion to developers of large-scale models
  - test robustness of linear solutions to potentially important effects of nonlinearities.